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RETROSPECT AND PROSPECT.

By H. E. SLAUGHT, University of Chicago.

During the year just closed the Editors of the MONTHLY have endeavored to make this journal occupy a unique position in this country. How well they have succeeded is not for them to say. However, it may be helpful to all concerned to review the situation briefly and to consider our plans for the coming year.

What we have tried not to do. We have tried not to encroach in any way upon the well-defined field of our excellent secondary school journals of mathematics. We have not explicitly considered secondary school problems or policies. We have not dwelt upon pedagogical matters pertaining directly to secondary mathematics. Nevertheless, it is by no means true that we are without interest in the secondary field. On the contrary, we have an interest that is far deeper than the mere formulation and discussion of committee reports or of pedagogical theories. We believe that the *teachers themselves* are by far the most important factor in any educational scheme, and that unless these teachers, at least those who are responsible for planning and directing the work, have sources of inspiration and power outside of themselves and independent of their daily routine, then the best of plans and of pedagogical rules of procedure will accomplish but little in actual practice. We conceive that the MONTHLY has a mission to perform in helping to supply such inspiration and power to those high school teachers who can be brought within the sphere of its influence. On this point see the remarks by T. G. Rodgers on page 32 of this issue, and also the recent action of California teachers of mathematics reported on page 36.

On the other hand we have tried not to encroach upon the field of the advanced scientific journals of mathematics. We have refrained from publishing articles of a highly specialized character, and indeed in all articles involving technical terms or discussions we have tried to insist upon such explanations and illustrations as would render the context intelligible to the average reader of fair attainment in the ordinary courses of college mathematics, including calculus, taken by

candidates for the Bachelor's degree. Nevertheless, it is by no means true that we are without interest in the higher, technical, mathematical field. On the contrary, we have an interest that is far more vital than the mere supplying of technical papers which can be read only by specialists. We believe that large numbers who would become active and effective in higher mathematical research are now lost to the cause simply by reason of the fact that there are no intermediate steps up which they can climb to these heights. We believe that the MONTHLY has a mission to perform in holding the interest of such persons by providing mathematical literature of a stimulating character that is within their range of comprehension, and by offering an appropriate medium for the publication of worthy papers which the more ambitious among them may produce.

What we have tried to do. Having in mind the principles stated above we have during 1913 supplied 325 pages of matter, exclusive of the index to Volume XX, distributed as follows: papers involving subjects of historical interest, 87 pages; papers involving general information concerning the progress of mathematics, such as meetings of associations, book reviews, notes and news, 57 pages; topics involving pedagogical considerations, especially with regard to subject matter, 37 pages; papers involving a minimum of mathematical technicalities and dealing with topics of wide interest, 56 pages; papers of a somewhat more technical character in which, however, we have tried to have the technical terms explained for the benefit of the general reader, 38 pages; problems proposed and solved and miscellaneous questions involving difficulties actually encountered by our readers, 50 pages. We have thus tried to maintain an appropriate balancing of matter so as to conserve the interests of all our readers.

What we desire to do during the coming year. During 1914 it will be our endeavor to maintain the standards already established and to improve upon the past in every way possible. In order to do this we need the coöperation and constructive criticism of all our friends. For example, a certain reader whose opinion is greatly appreciated thinks that we should have more papers on topics in applied mathematics, and he immediately backs up his opinion by sending us a contribution which will appear in the March issue. That is what we mean by *coöperation.* The editors have no possible interest in this undertaking which should not appeal directly to every one who is really concerned for the development of mathematics in this country. Their responsibilities and burdens are self-imposed and without emolument, save for the satisfaction which may accrue from aiding in a cause in which they heartily believe. It is their ambition to make the MONTHLY render genuine service to every teacher of courses in college mathematics in this country, whether in academy, high school, normal school, college, or university; to stimulate to higher endeavor every student of mathematics, whether in school or not, who may be attracted by the papers, problems, questions or discussions published in the MONTHLY; and to win and hold the coöperation of all who can in any department render assistance in carrying out these plans. In particular, every one who reads these lines may at once give important assistance by bringing the MONTHLY to the attention of those who

may not know about it. The subscription list was doubled in 1913. If it should be doubled again in 1914, the journal would become self-sustaining. **It can be done.**

THE TACTICAL PROBLEM OF STEINER.

By W. H. BUSSEY, University of Minnesota.

NOTE BY THE EDITORS.—This article illustrates the reference in the editorial of this issue concerning "papers of a somewhat more technical character in which, however, we have tried to have the technical terms explained for the benefit of the general reader." Professor Bussey has met this request most admirably.

The study of tactical configurations known as triple systems had its origin in two related problems proposed independently by T. P. Kirkman[1] and J. Steiner.[2] Kirkman's problem is to arrange fifteen school girls in parties of three for seven consecutive days' walk so that every two of the girls walk together once and only once during the seven days. There is a good account of the history of the problem with several methods of solution in Ball's *Mathematical Recreations and Essays,* 5th edition, Chapter 9.

An arrangement of a number of elements in sets of three so that every set of two is contained in one and only one set of three is called a triple system. The sets of three are called triples or triads. The problem of the fifteen school girls involves a triple system of 15 elements and 35 triads. The simplest triple system is the following well-known one of 7 elements and 7 triads. The digits 0, 1, 2, 3, 4, 5, 6 are the elements and the columns are the triads.

0	1	2	3	4	5	6
1	2	3	4	5	6	0
3	4	5	6	0	1	2

The seven elements of this triple system can be arranged in sets of four, called tetrads, so that no triad is contained in a tetrad and so that every set of three which is not a triad is contained in one and only one tetrad. The arrangement is as follows.

0	1	2	3	4	5	6
1	2	3	4	5	6	0
2	3	4	5	6	0	1
5	6	0	1	2	3	4

It is well known that the nine points of inflexion of a plane cubic curve lie by threes on twelve straight lines. Four lines pass through each point of inflexion. Any two of the nine points thus determine a third, and the nine points form a

[1] *The Lady's and Gentleman's Diary,* 1850.

[2] *Journal für die reine und angewandte Mathematik,* Vol. 45, pp. 181–182.

triple system with twelve triads. The triple system of nine elements may be written as follows.

0	0	0	0	1	1	1	2	2	2	3	6
1	3	4	5	3	4	5	3	4	5	4	7
2	6	7	8	7	8	6	8	6	7	5	8

These triple systems of seven and nine elements are important in the theory of equations of the 7th and 9th degrees.[1]

The problem proposed by Steiner is as follows. It was suggested to him by an investigation of the configuration of the 28 double tangents of a plane quartic curve.

For what values of n is it possible to arrange n elements in sets of three, called triples or triads, so that every set of two elements is contained in one and only one triad? If n is a number for which there is such an arrangement in triads, are there other arrangements which cannot be obtained from it by a mere permutation of the elements? When such an arrangement has been made, is it possible to arrange the n elements in sets of four, called tetrads, so that no triad is contained in a tetrad and so that every set of three which is not a triad is contained in one and only one tetrad? When such an arrangement in tetrads has been made, is it possible to arrange the n elements in sets of five, called pentads, so that no triad or tetrad is contained in a pentad, and so that every set of four which is not a tetrad and does not contain a triad is contained in one and only one pentad? When these successive arrangements have been made, up to and including an arrangement in *k-ads*, is it possible to arrange the n elements in sets of $k + 1$, called $(k + 1)$-*ads*, so that no *l-ad*, $l \leq k$, is contained in a $(k + 1)$-*ad*, and so that every set of k elements which is not a *k-ad* and does not contain an *l-ad*, $l < k$, is contained in one and only one $(k + 1)$-*ad?*

The part of the problem that relates to triads has been completely solved.[2] This means that all of Steiner's questions have been answered. It does not mean that the last word on triple systems has been said or that all problems connected with triple systems have been solved.

Every pair of elements of a triple system determines a triad. The number of pairs that can be chosen from n elements is $\frac{1}{2}n(n - 1)$. But each triad is determined by any one of three pairs. Therefore, to obtain the number of triads, divide the number of pairs by three. The result is $\frac{1}{6}n(n - 1)$. The total number of elements in the triple system, counting duplicates, is three times the number of triads, namely $\frac{1}{2}n(n - 1)$. Therefore each of the n elements occurs $\frac{1}{2}(n - 1)$ times. This number must be an integer, and therefore n must be an odd number. It must therefore be of one of the forms $6m + 1, 6m + 3, 6m + 5$. But $\frac{1}{6}n(n - 1)$ must also be an integer. This can happen when n is $6m + 1$ or $6m + 3$, but not when n is $6m + 5$. Therefore a necessary condition that n elements can be ar-

[1] See Netto's *Theory of Substitutions* (English translation by F. N. Cole), pp. 229–239, and Netto's *Vorlesungen über Algebra*, Vol. 2, pp. 460–480.

[2] See *Encyclopédie des Sciences Mathématiques*, Vol. 1, p. 80.

ranged to form a triple system is that n be of the form $6m + 1$ or $6m + 3$. It has been proved that this condition is also sufficient. When $n = 7$ or 9, there is essentially only one arrangement of the n elements in triads. Any two arrangements differ only in notation. When $n = 13$, there are two and only two essentially different triple systems. When n is any number of the form $6m + 1$ or $6m + 3$ and is greater than 13, there are at least two essentially different triple systems.[1] The two triple systems of 13 elements are compared in two recent papers by F. N. Cole[2] and H. S. White.[3]

This paper has to do primarily with the part of the Steiner problem that relates to tetrads, pentads, etc. If an arrangement of n elements in triads and tetrads is possible, the number of tetrads can be counted as follows. Every tetrad is determined by a set of three which is not a triad. The total number of sets of three that can be chosen from n elements is $\frac{1}{6}n(n - 1)(n - 2)$. The number of these which are triads has been found to be $\frac{1}{6}n(n - 1)$. The difference between these two numbers, namely $\frac{1}{6}n(n - 1)(n - 3)$, is the number of sets of three which determine tetrads. But each tetrad can be determined by three elements in as many ways as there are combinations of four things three at a time, namely in four ways. Therefore, to get the number of tetrads, divide by four. The result is $\frac{1}{24}n(n - 1)(n - 3)$. When $n = 7$, this number is also equal to 7. An arrangement of seven elements in seven tetrads has already been given.

If there is an arrangement of n elements in triads, tetrads, pentads, etc., the number of *k-ads* for $k = 3, 4, 5$, etc., is given by the formula

$$N_k = \frac{1}{k!}n(n - 1)(n - 3)(n - 7) \cdots (n - [2^{k-2} - 1]),$$

which was given by Steiner without proof. It can be proved as follows. Let $a_1, a_2, a_3, \cdots, a_{k-1}$ be a set of $k - 1$ elements that is not a $(k - 1)$-*ad* and does not contain an *l-ad*, $l < k - 1$. Such a set determines a *k-ad.* The number of such sets is k times the number of *k-ads*, namely kN_k, because the same *k-ad* is determined by any one of k different sets of $k - 1$ elements each. Every $(k + 1)$-*ad* is determined by a set of k elements that is not a *k-ad* and does not contain an *l-ad*, $l < k$. Any $k - 1$ elements of such a set constitute a set $a_1, a_2, a_3, \cdots, a_{k-1}$ of the kind mentioned above. Therefore, to get a set of k elements to determine a $(k + 1)$-*ad*, adjoin to these an element a_k which is such that it will not combine with any of the others to form an *l-ad* for any $l \leqq k$. In choosing a_k, the following elements must be avoided.

1. The ${}_{k-1}C_1$ elements $a_1, a_2, a_3, \cdots, a_{k-1}$.
2. The ${}_{k-1}C_2$ elements that form triads with the elements $a_1, a_2, a_3, \cdots, a_{k-1}$ taken in pairs.

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[1] For proofs of these statements see Moore, *Concerning Triple Systems, Mathematische Annalen,* Vol. 43, pp. 271–285, and Netto, *Vorlesungen über Algebra,* Vol. 2, p. 474.

[2] *Transactions of the American Mathematical Society,* Vol. 14 (1913), pp. 1–5.

[3] *Transactions of the American Mathematical Society,* Vol. 14 (1913), pp. 6–13.

3. The ${}_{k-1}C_3$ elements that form tetrads with the elements $a_1, a_2, a_3, \cdots, a_{k-1}$ taken in threes.

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$k - 1$. The ${}_{k-1}C_{k-1}$ elements that form *k-ads* with the elements $a_1, a_2, a_3, \cdots, a_{k-1}$ taken $k - 1$ at a time.

The total number of elements to be avoided is therefore

$$ {}_{k-1}C_1 + {}_{k-1}C_2 + {}_{k-1}C_3 + \cdots + {}_{k-1}C_{k-1}, $$

which is the number of combinations of $k - 1$ things taken any number at a time, and is equal to $2^{k-1} - 1$.[1] The number of ways in which the a_k can be chosen is therefore $n - (2^{k-1} - 1)$, and the number of ways in which k elements can be chosen to determine a $(k + 1)$-*ad* is apparently $kN_k[n - (2^{k-1} - 1)]$. But the set of k elements has been obtained by adjoining an element a_k to a certain sub-set $a_1, a_2, a_3, \cdots, a_{k-1}$. It can be obtained in the same way by adjoining an element to any other sub-set of $k - 1$ of the same k elements. These sub-sets are ${}_kC_{k-1} = k$ in number. Therefore the count is k times as large as it ought to be; and the number of ways in which a set of k elements can be chosen to determine a $(k + 1)$-*ad* is $N_k[n - (2^{k-1} - 1)]$. But the same $(k + 1)$-*ad* is determined by any k of its elements, that is in ${}_{k+1}C_k = k + 1$ ways. Therefore the total number of $(k + 1)$-*ads* is $N_k\left[\dfrac{n - (2^{k-1} - 1)}{k + 1}\right]$. In other words, to obtain the number of $(k + 1)$-*ads*, multiply the number of *k-ads* by $\dfrac{n - (2^{k-1} - 1)}{k + 1}$. The number of tetrads has already been found to be $\dfrac{1}{4!}n(n - 1)(n - 3)$. Therefore the number of pentads is $\dfrac{1}{5!}n(n - 1)(n - 3)(n - 7)$; and the number of *k-ads* for any value of k is that given by Steiner's formula.

In the *Bulletin of the American Mathematical Society*, Vol. 16, pp. 19–22, the author of this paper proved that when n is a number of the form $2^j - 1$ there exists an arrangement of n elements in triads, tetrads, pentads, etc., up to and including $(j + 1)$-*ads*. That paper made use of finite projective geometries of many dimensions.[2] The present paper gives the solution in a more simple way which makes no use of geometry. It not only proves that the arrangement is possible, but also gives a method by which the arrangement can easily be written down in any particular case. The paper makes use of the theory of linear dependence in a finite field.

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[1] See Fine's *College Algebra*, § 770.

[2] See Veblen and Bussey, "Finite Projective Geometries," *Transactions of the American Mathematical Society*, Vol. 7, pp. 241–259.

Note.—M. G. Brunel, in *Procès-Verbaux, Soc. des Sciences de Bordeaux,* 1896–1897, pp. 37–41, gave an arrangement of 9 elements in triads and tetrads; one of 7 elements in triads and tetrads; and one of 15 elements in triads, tetrads, and pentads. He stated that the method which he used in the last two cases could be extended to the case of $2^j - 1$ elements. His paper gave merely the results in the three cases ($n = 7, 9, 15$) without any suggestion of method.

THE FINITE ALGEBRA OF ORDER p.

Any system of s distinct symbols or marks which can be combined by four operations that satisfy the same formal laws as the four fundamental operations of algebra, addition, subtraction, multiplication, and division, so that when the marks are so combined the result of the operations is in every case unique and belongs to the given system of marks, are said to constitute a "finite algebra" or "field" of order s.

In the theory of numbers the integers $0, 1, 2, \cdots, (p - 1)$ are said to constitute a system of "least residues," modulo p. No two of them are congruent to each other, and every other integer is congruent to some one of them, modulo p. The sum of two of them, say a and b, is congruent to some one of them, say c. Then c is said to be the "sum of a and b modulo p." The operation of finding c when a and b are given is called "addition modulo p." Similarly "subtraction modulo p" and "multiplication modulo p" are defined. It is proved in the theory of numbers that, when p is a prime, there is one and only one least residue c of p, such that $ac \equiv b$, modulo p, where b is any least residue of p and a is any one except zero. c is called the quotient of b divided by a modulo p, and the operation of finding c when b and a are given is called "division modulo p." Methods of finding c in any given case are given in books on the theory of numbers. It is also proved in the theory of numbers that these operations which are called addition, subtraction, multiplication, and division, modulo p, satisfy the same formal laws as the corresponding operations of ordinary algebra. Therefore the least residues modulo p, where p is a prime, constitute a "finite algebra" or "field" of order p.

LINEAR DEPENDENCE IN THE FIELD OF ORDER p.

In the field of order p, m sets of n marks each

$$(A) \qquad \begin{array}{ccccc} x_1', & x_2', & x_3', & \cdots, & x_n' \\ x_1'', & x_2'', & x_3'', & \cdots, & x_n'' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1^{[m]}, & x_2^{[m]}, & x_3^{[m]}, & \cdots, & x_n^{[m]} \end{array}$$

are said to be linearly dependent when there exist in the field m marks c', c'', c''', $\cdots, c^{[m]}$, not all zero, such that

$$c'x_j' + c''x_j'' + c'''x_j''' + \cdots + c^{[m]}x_j^{[m]} \equiv 0, \text{ mod. } p \quad (j = 1, 2, 3, \cdots, n).$$

If this is not the case, the sets are said to be linearly independent.

This definition is analogous to the definition of linear dependence in ordinary algebra.[1] The proofs of the following theorems are almost word for word like the proofs of the corresponding theorems in ordinary algebra, and for that reason are not given here.

THEOREM 1. Two sets of n marks each, not all zero, are linearly dependent when and only when they are proportional.

THEOREM 2. If there exist among m sets of marks a smaller number of sets which are linearly dependent, then the m sets are linearly dependent.

THEOREM 3. If any one of the m sets of marks consists wholly of zeros, the m sets are linearly dependent.

THEOREM 4. A necessary and sufficient condition that the m sets of marks (A) be linearly dependent when $m \leqq n$ is that all the m-rowed determinants of the matrix

$$\begin{Vmatrix} x_1' & x_2' & x_3' & \cdots & x_n' \\ x_1'' & x_2'' & x_3'' & \cdots & x_n'' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1^{[m]} & x_2^{[m]} & x_3^{[m]} & \cdots & x_n^{[m]} \end{Vmatrix}$$

be congruent to zero.

THEOREM 5. m sets of n marks each are always linearly dependent when $m > n$.

LINEAR DEPENDENCE IN THE FIELD OF ORDER 2.

In the field of order 2, the only marks are 0 and 1, and therefore, by Theorem 1, two sets of n marks each, not all zero, are linearly dependent when and only when they are identical.

Consider m different sets of n marks each, not all zero, in the field of order 2, $m \leqq n$,

$$(B) \qquad \begin{matrix} x_1', & x_2', & x_3', & \cdots, x_n' \\ x_1'', & x_2'', & x_3'', & \cdots, x_n'' \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{[m]}, & x_2^{[m]}, & x_3^{[m]}, & \cdots, x_n^{[m]} \end{matrix}$$

such that the m sets are linearly dependent and such that no q of the sets are linearly dependent for any $q < m$. Then there are in the field m marks c', c'', $c''', \cdots, c^{[m]}$, not all zero, such that

$$c'x_j' + c''x_j'' + c'''x_j''' + \cdots + c^{[m]}x_j^{[m]} \equiv 0, \bmod 2 \quad (j = 1, 2, 3, \cdots, n).$$

[1] For an account of the theory of linear dependence in ordinary algebra, see Bôcher's "Introduction to Higher Algebra," Chapter III.

No one of the c's can be zero; for if one of them were, say $c^{[m]}$, this relation would become

$$c'x_j' + c''x_j'' + c'''x_j''' + \cdots + c^{[m-1]}x_j^{[m-1]} \equiv 0, \text{ mod } 2 \quad (j = 1, 2, 3, \cdots, n),$$

where the c's are not all zero, which would prove that the first $m - 1$ sets are linearly dependent. But this is contrary to the hypothesis about the sets (B). Therefore every one of the c's must be 1, and the following theorem is proved.

THEOREM 6. If m sets of n marks each, not all zero, in the field of order 2, $m \leqq n$, are linearly dependent, and if no q of them, $q < m$, are linearly dependent, then

$$x_j' + x_j'' + x_j''' + \cdots + x_j^{[m]} \equiv 0, \text{ mod } 2 \quad (j = 1, 2, 3, \cdots, n).$$

THEOREM 7. Any one of the marks $x_j^{[i]}$, $[i = 1, 2, \cdots, m]$, of Theorem (6), is congruent to the sum of all the others, modulo 2.

This is because, in the algebra of integers modulo 2, the mark -1 is congruent to the mark $+1$, and consequently any term of a congruence can be transposed from one side of the congruence to the other.

Let the following be $m - 1$ linearly independent sets of marks of the field of order 2.

$$\begin{array}{llll}
x_1', & x_2', & x_3', & \cdots, x_n' \\
x_1'', & x_2'', & x_3'', & \cdots, x_n'' \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
x_1^{[m-1]}, & x_2^{[m-1]}, & x_3^{[m-1]}, & \cdots, x_n^{[m-1]}
\end{array}$$

Adjoin to these a new set $x_1^{[m]}, x_2^{[m]}, \cdots, x_n^{[m]}$, where $x_j^{[m]}$, $(j = 1, 2, 3, \cdots, n)$, is the mark of the field of order 2 determined by the congruence

$$x_j^{[m]} \equiv x_j' + x_j'' + x_j''' + \cdots + x_j^{[m-1]}, \text{ mod } 2, \tag{1}$$

which can also be written in the form

$$x_j' + x_j'' + x_j''' + \cdots + x_j^{[m-1]} + x_j^{[m]} \equiv 0, \text{ mod } 2.$$

This last relation proves that the m sets are linearly dependent. It can be proved that no $m - 1$ of the m sets are linearly dependent. The proof is as follows. The first $m - 1$ sets are linearly independent by hypothesis, and therefore, by Theorem 4, at least one of the $(m - 1)$-rowed determinants of the matrix

$$\left\| \begin{array}{lllll}
x_1' & x_2' & x_3' & \cdots & x_n' \\
x_1'' & x_2'' & x_3'' & \cdots & x_n'' \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
x_1^{[m-1]} & x_2^{[m-1]} & x_3^{[m-1]} & \cdots & x_n^{[m-1]}
\end{array} \right\|$$

is not congruent to zero. It can be assumed without loss of generality that the sets have been so arranged that the determinant of the first $m - 1$ columns is one not congruent to zero; that is, the determinant

$$(C) \qquad \begin{vmatrix} x_1' & x_2' & x_3' & \cdots & x_{m-1}' \\ x_1'' & x_2'' & x_3'' & \cdots & x_{m-1}'' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1^{[m-1]} & x_2^{[m-1]} & x_3^{[m-1]} & \cdots & x_{m-1}^{[m-1]} \end{vmatrix}$$

is not congruent to zero. Now consider $m - 1$ of the m sets including the m-th set, and suppose the one omitted to be the first set. The necessary and sufficient condition that they be linearly dependent is that all the $(m - 1)$-rowed determinants of the matrix

$$\begin{Vmatrix} x_1'' & x_2'' & x_3'' & \cdots & x_n'' \\ x_1''' & x_2''' & x_3''' & \cdots & x_n''' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1^{[m-1]} & x_2^{[m-1]} & x_3^{[m-1]} & \cdots & x_n^{[m-1]} \\ x_1^{[m]} & x_2^{[m]} & x_3^{[m]} & \cdots & x_n^{[m]} \end{Vmatrix}$$

be congruent to zero. The $(m - 1)$-rowed determinant of the first $m - 1$ columns is

$$(D) \qquad \begin{vmatrix} x_1'' & x_2'' & x_3'' & \cdots & x_{m-1}'' \\ x_1''' & x_2''' & x_3''' & \cdots & x_{m-1}''' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1^{[m]} & x_2^{[m]} & x_3^{[m]} & \cdots & x_{m-1}^{[m]} \end{vmatrix}$$

Replace each element of the last row by its value as given by relation (1), and the determinant becomes

$$\begin{vmatrix} x_1'' & x_2'' & x_{m-1}'' \\ x_1''' & x_2''' & x_{m-1}''' \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ x_1^{[m-1]} & x_2^{[m-1]} & x_{m-1}^{[m-1]} \\ (x_1'+x_1''+\cdots+x_1^{[m-1]}), & (x_2'+x_2''+\cdots+x_2^{[m-1]}), & \cdots (x_{m-1}'+x_{m-1}''+\cdots+x_{m-1}^{[m-1]}) \end{vmatrix}$$

By the well-known theorem about the addition of determinants, this determinant is equal to the sum of $m - 1$ determinants, the first of which is

$$\begin{vmatrix} x_1'' & x_2'' & \cdots & x_{m-1}'' \\ x_1''' & x_2''' & \cdots & x_{m-1}''' \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{[m-1]} & x_2^{[m-1]} & \cdots & x_{m-1}^{[m-1]} \\ x_1' & x_2' & \cdots & x_{m-1}' \end{vmatrix}$$

The others are all congruent to zero, because in each one of them two rows are identical. The one determinant is not congruent to zero because it differs from the determinant (C) only in the order of its rows. Therefore the determinant (D) is not congruent to zero; and the $m - 1$ sets under consideration are not linearly dependent. It has now been proved that the $m - 1$ sets obtained by omitting the first of the m sets are linearly independent. The argument holds good when the omitted set is not the first one but any other one. This completes the proof that no $m - 1$ of the m sets are linearly dependent. By Theorem 2, it follows that no q of the m sets, $q < m$, are linearly dependent. The result may be stated as the following theorem.

THEOREM 8. Any $m - 1$ sets of n marks each, not all zero, in the field of order 2, which are linearly independent determine an m-th set so that the m sets are linearly dependent and so that no q of them, $q < m$, are linearly dependent.

APPLICATION TO STEINER'S PROBLEM.

Consider the $2^j - 1$ different elements of the form $(x_1, x_2, x_3, \cdots, x_j)$, each x being a mark of the field of order 2, that is, either 0 or 1, the element $(0, 0, 0, \cdots, 0)$ being excluded. Any two of them are linearly independent because they are different. Any three which are linearly dependent are said to constitute a triad. By Theorem 8, any two elements determine a third to form a triad. If the two elements are $(x_1, x_2, x_3, \cdots, x_j)$ and $(y_1, y_2, y_3, \cdots, y_j)$, the third is $(x_1 + y_1, x_2 + y_2, x_3 + y_3, \cdots, x_j + y_j)$, where the + signs mean addition modulo 2. Also by Theorem 8, any three elements which are linearly independent, and therefore do not form a triad, determine a fourth element so that the four are linearly dependent but no three of them are linearly dependent. Such a set of four is called a tetrad. Any set of four elements which are linearly independent is not a tetrad and, by Theorem 2, does not contain a triad. Such a set of four determines a fifth element, by Theorem 8, such that the five are linearly dependent but also such that no four or three are linearly dependent. Such a set of five is called a pentad. Any set of k elements which are linearly independent is not a k-*ad* and, by Theorem 2, does not contain an l-*ad*, $l < k$. The k elements determine a $(k + 1)$-st element, by Theorem 8, such that the $k + 1$ elements are linearly dependent but also such that no l of them, $l < k + 1$, are linearly de-

pendent. Such a set of $k + 1$ elements is called a $(k + 1)$-*ad*. This arrangement in triads, tetrads, etc., is just the arrangement called for in the Steiner problem. There is no arrangement of $2^j + 1$ elements in *k-ads* for $k > j + 1$ because, by Theorem 5, there is no set of $j + 1$ linearly independent elements. Furthermore, the Steiner formula gives $N_k = 0$ when $k > j + 1$. But there is an arrangement in *k-ads* for every $k \leqq j + 1$.

As an example of the foregoing theory, the arrangement of seven elements in triads and tetrads is worked out as follows. In this case $j = 3$, and the $2^j - 1 = 7$ elements are

$$(001)\quad (010)\quad (101)\quad (011)\quad (111)\quad (110)\quad (100).$$

For convenience let these be denoted by the letters a, b, c, d, e, f, g, respectively. Any two elements (x_1, x_2, x_3) and (y_1, y_2, y_3) determine the third element $(x_1 + y_1, x_2 + y_2, x_3 + y_3)$, where the $+$ signs mean addition modulo 2. The three form a triad. Thus the elements a and b determine the element (001), which is the element d. Therefore one triad of the system is *abd*. Similarly all the other triads can be worked out. The complete arrangement is as follows, the columns being triads.

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \\ b & c & d & e & f & g & a \\ d & e & f & g & a & b & c \end{array}$$

Let (x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3) denote any three elements that do not form a triad. They determine as the fourth element to form a tetrad the element $(x_1 + y_1 + z_1, x_2 + y_2 + z_2, x_3 + y_3 + z_3)$, where the $+$ signs mean addition modulo 2. Thus the three elements a (001); b (010); and c (101) determine the element (110), which is the element f. Therefore one tetrad of the system is *abcf*. The others can be found in the same way. The complete arrangement is as follows, the columns being the tetrads.

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \\ b & c & d & e & f & g & a \\ c & d & e & f & g & a & b \\ f & g & a & b & c & d & e \end{array}$$

This arrangement in triads and tetrads is, apart from notation, the same as the one given earlier in the paper.

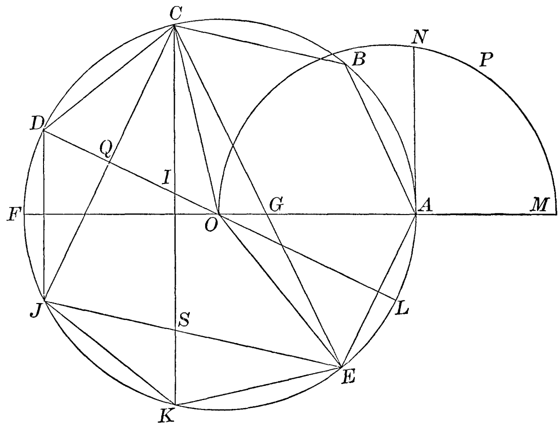
The following table gives the number of triads, tetrads, etc., when $n = 7, 15, 31, 63$.

n.	N_3.	N_4.	N_5.	N_6.	N_7.	N_8.
7	7	7	0	0	0	0
15	35	105	168	0	0	0
31	155	1,085	5,208	13,888	0	0
63	651	9,765	109,368	974,944	3,999,744	0

A GEOMETRICAL DISCUSSION OF THE REGULAR INSCRIBED HEPTAGON.

By J. Q. McNatt, Florence, Colo.

To calculate the length of the side of a regular inscribed heptagon when the radius is unity, suppose, in the accompanying figure, that the circumference is divided into seven equal parts at the points A, B, C, D, J, K, and E.



Draw the diameter DL and the chords JC, JE, CK, and CE and let CK cut JE at S.

Let h be the length of the side of the regular inscribed heptagon. It is proposed to compute h in terms of the radius as a unit. We have $DL : DC :: DC : DQ$, or in terms of the radius and h, $2 : h :: h : DQ$. That is, $DQ = h^2/2$. Moreover, $QL \times DQ = QC \times JQ$, and $QC = JQ$, or in terms of h we get

$$JQ = \sqrt{\left(2 - \frac{h^2}{2}\right)\left(\frac{h^2}{2}\right)} = \tfrac{1}{2}h\sqrt{4 - h^2}.$$

(1) Now $JE = JC = h\sqrt{4 - h^2}$, and $KE = SE$ since the two triangles KCE and KES are equiangular and hence similar, and since KCE is isosceles. Then in terms of h we find (2) $JS = JE - SE$, or $JS = h\sqrt{4 - h^2} - h$.

Now let DL cut KC at I and draw the radius CO. Then the two triangles OCD and CDI are equiangular and similar, and OCD is isosceles.

Then $CO : DC :: DC : DI$ or in terms of h, $1 : h :: h : DI$, from which $DI = h^2$. Also $DL - DI = IL = 2 - h^2$. Again, triangle DCI being isosceles, $DC = CI$, or $CI = h$. By geometry $IL \times DI = CI \times IK$, or in terms of h we have,

$$(2 - h^2)(h^2) = (h)(IK), \quad \text{or} \quad IK = 2h - h^3,$$

but $CI = h$, so that $KC = 3h - h^3$.

As the triangles KES and KCE are similar isosceles triangles, we have

$$CK : KE :: KE : SK,$$

or in terms of h,

(3) $$3h - h^3 : h :: h : SK,$$

from which

$$SK = \frac{h}{3 - h^2}.$$

In the isosceles triangle JCS, $CS = JC = h\sqrt{4 - h^2}$ as shown above in (1). By geometry $CS \times SK = JS \times SE$, or in terms of h, using (1), (2) and (3) above,

(4) $$(h\sqrt{4 - h^2})\left(\frac{h}{3 - h^2}\right) = (h\sqrt{4 - h^2} - h)(h).$$

Simplifying equation (4) we find

$$8 - 6h^2 + h^4 = (3 - h^2)\sqrt{4 - h^2}.$$

Factoring, dividing both members by $\sqrt{4 - h^2}$, squaring, and transposing, we get

(5) $$7 - 14h^2 + 7h^4 - h^6 = 0,$$

which we call *The Heptagon Cubic*. Solving (5) by Horner's method we find,

$$h^2 = .7530203962821 \cdots,$$

or

$$h = .86776748 \cdots,$$

which is the side of the regular heptagon inscribed in a circle of radius unity.

Approximate Construction of a Heptagon.

From the similar isosceles triangles OEA and EAG, we get

(6) $$OE : EA :: EA : GA,$$

or

$$1 : h :: h : GA,$$

from which

$$GA = h^2,$$

but

$$h^2 = .75 +, \text{ or } \tfrac{3}{4}OA.$$

Produce the line FA indefinitely and lay off on it the point G, so that

$$GA = \tfrac{3}{4}OA.$$

Then mark the point M, so that $AM = GA$.

With OM as a diameter construct the semicircle OPM and at the point A erect a perpendicular AN, which will be a mean proportional between OA and AM, that is, between OE and GA, so that by (6) $AN = EA$.

Then with a chord equal to the line AN lay off the points of the seven sided polygon, A, B, C, D, etc., which will be a regular inscribed heptagon, to a very close approximation.

A THEOREM IN NUMBER THEORY CONNECTED WITH THE BINOMIAL FORMULA.

By D. N. LEHMER, University of California.

The theorem is: *In the expansion of* $(P + Q)^n$, *where* P *and* Q *as well as* n *are integers, there will be a pair of consecutive equal terms when, and only when,* $n \equiv -1 \bmod (P + Q)/\delta$, *where* δ *is the greatest common divisor of* P *and* Q.

Equating two consecutive terms we have

$$\frac{n(n-1)(n-2)(n-3)\cdots(n-k)}{(k+1)!}P^{n-k-1}Q^{k+1} = \frac{n(n-1)(n-2)\cdots(n-k-1)}{(k+2)!}P^{n-k-2}Q^{k+2}.$$

Cancelling and multiplying across,

$$P(k+2) = (n-k-1)Q.$$

If now $P = \delta P'$ and $Q = \delta Q'$ where P' and Q' are relatively prime,

$$P'(k+2) = (n-k-1)Q',$$

whence

$$nQ' = 2P' + Q' + k(P' + Q')$$

or

$$(n+1)Q' \equiv 2(P' + Q') \bmod (P' + Q'),$$

that is

$$(n+1)Q' \equiv 0 \bmod (P' + Q'),$$

and since Q' is prime to $P' + Q'$,

$$n + 1 \equiv 0 \bmod (P' + Q').$$

That is

$$n \equiv -1 \bmod (P + Q)/\delta.$$

More generally if $n \equiv -1 \bmod (P + Q)/\delta$ there will be two consecutive terms in the expansion having the ratio β/α, δ being the greatest common divisor of αP and βQ.

AN APPLICATION OF PARTIAL DERIVATIVES TO THE ELLIPSE.

By M. O. TRIPP, Yonkers, N. Y.

The object of this article is to show how we may make use of partial derivatives in graphing and discussing the properties of a given ellipse.

Let us take the equation of an ellipse, in rectangular coördinates, in the general form

$$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \quad (h^2 - ab < 0),$$

from which we obtain

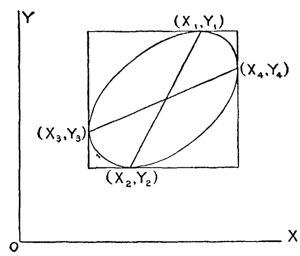
$$\frac{\partial f(x, y)}{\partial x} = 2ax + 2hy + 2g, \quad \text{and} \quad \frac{\partial f(x, y)}{\partial y} = 2hx + 2by + 2f.$$

Making use of the formula

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f(x, y)}{\partial x}}{\dfrac{\partial f(x, y)}{\partial y}}$$

we have, for the slope of the tangent at any point on the ellipse,

$$\frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f}. \tag{1}$$



At the highest and lowest points of the curve, namely, (x_1, y_1) and (x_2, y_2) respectively, $dy/dx = 0$, since the geometrical tangents at these points are parallel to the x-axis. Hence we observe from (1) that the equation

$$ax + hy + g = 0 \tag{2}$$

is satisfied by the coördinates of the points (x_1, y_1) and (x_2, y_2), since the denominator in the right member of (1) cannot become infinite for these values. Thus it follows that (2) is the equation of the diameter joining (x_1, y_1) and (x_2, y_2); it is the equation of the diameter bisecting the chords parallel to the axis of x.

At the points farthest to the left and right on the curve, namely, (x_3, y_3) and (x_4, y_4) respectively, $dx/dy = 0$, since the geometrical tangents at these points are perpendicular to the x-axis.
Hence

$$hx + by + f = 0, \tag{3}$$

since the numerator of the right member of (1) cannot become infinite for the values considered.

Hence (3) represents the equation of the diameter joining (x_3, y_3) and (x_4, y_4); (3) is the equation of the diameter bisecting chords parallel to the axis of y.

It follows that the coördinates of the center of the ellipse are given by solving simultaneously the equations (2) and (3), or what amounts to the same thing, the equations

$$\frac{\partial f(x, y)}{\partial x} = 0 \tag{4}$$

and

$$\frac{\partial f(x, y)}{\partial y} = 0. \tag{5}$$

It is obvious that all of the methods described above may be applied to any conic

$$ax^2 + 2bxy + cy^2 + 2gx + 2fy + c = 0.$$

If any failure results, it is certain that the conic is a special form for which the corresponding concept does not exist. Thus the usual rule for determining whether (6) is or is not a parabola results from the test whether (4) and (5) have or have not a common solution.

The tests for highest and lowest points apply, with obvious restrictions, to any curve whatever.

BOOK REVIEWS.

UNDER THE DIRECTION OF W. H. BUSSEY.

Algebra, First Course. By EDITH LONG and W. C. BRENKE. The Century Co., New York, 1913. viii+283 pages.

This is the first of a series of texts of correlated mathematics for secondary schools and, in particular, it is the first year's work of a two years' course in algebra and geometry. In its preparation, the authors had in mind two purposes. The first was, in their own words, "to vivify the treatment of algebra by a systematic correlation with geometry." The other was to present the matter in a more simple narrative style than is done in most texts.

To carry out the first purpose, the authors have introduced some of the simpler theorems of geometry and the measurement of geometric quantities as well as some of the fundamental concepts of physics. The proofs of the theorems are in the nature of experimental verification. The introduction of such new matter will undoubtedly stimulate the interest of many students but it is a question if the appeal will be as wide as some have thought. It is evident that a class using this text cannot become as proficient in handling ordinary algebraic operations as is possible with other books. This is more than offset by the gain in the second year's work.

The topics treated are generally well and clearly handled. Where possible, the authors have made use of graphic methods to illustrate the principles. The simple style lends itself well to explanations. In definitions there is danger that it will lead to indistinct or even false conceptions, as when an angle is described as a figure formed by two straight lines starting from the same point.

The authors have requested that teachers using the book omit as little of the text as possible. This request merely points out the fact that the teacher as well as the student will need to be educated in its use.

R. R. SHUMWAY.

An Elementary Treatise on Calculus. By WILLIAM S. FRANKLIN, BARRY MACNUTT and ROLLIN L. CHARLES. Published by the authors, South Bethlehem, Pa., 1913. x+253+41 pages. $2.00.

This book, styled in the subtitle "a textbook for colleges and technical schools," is a noteworthy contribution to the texts carrying out the "Perry movement" in this country. The outstanding feature of the book is expressed in the preface in the purpose of the authors "to lead the student to a clear understanding of principles" and "to develop the subject as simply and as directly as possible." And yet it is exactly at this point that the reviewer's chief criticism is raised; this is in regard to the introduction of infinitesimals as indicated by the following quotation from page 23 of the text:

"In many cases, however, it is desirable to find the limiting relation between Δy and Δx when Δx approaches zero *without deriving an expression for* $\Delta y/\Delta x$. For example let $y = ax^3$. Then

$$\Delta y = 3ax^2 \cdot \Delta x + 3ax(\Delta x)^2 + a(\Delta x)^3. \tag{1}$$

Both members of this equation approach zero when Δx approaches zero, and it would therefore seem rather difficult to determine the limiting relation between Δy and Δx (when Δx approaches zero) from this equation as it stands. But when Δx is made as small as you please, then $(\Delta x)^2$ is infinitely smaller than Δx, and $(\Delta x)^3$ is infinitely smaller than $(\Delta x)^2$. For example, let Δx be a millionth of a unit, ... Therefore the terms $3ax(\Delta x)^2$ and $a(\Delta x)^3$ become more and more nearly negligible in comparison with $3ax^2\Delta x$ as Δx grows smaller and smaller in equation (1). The *limiting relation* between Δy and Δx may be found by writing dx and dy for Δx and Δy to indicate that we have proceeded to the limit, and by dropping every term which contains the square or any higher power of Δx (or the square or any higher power of Δy). This gives

$$dy = 3ax^2 \cdot dx. \tag{2}$$

In this equation dy and dx are as small as you please and they are called *infinitesimals;* . . . ."

In their preface the authors disclaim any intention of taking sides in the old controversy about the method of limits and that of infinitesimals, and they acknowledge there that the treatment here given is fallacious. The question then arises fairly and squarely whether they as textbook makers are justified, on grounds of simplicity or practical purposes or for any such reason, in using an analysis so inexact in the presentation of the fundamental ideas of the calculus. The writer is frank to say that in his judgment they are not justified. Beginners in calculus grasp the technique of differentiating and integrating rather than the analytic work involved in the proofs of the formulas and in the opening discussions, which latter should give *exact* notions of the calculus; and it surely does not sharpen these immature minds to read a text which with only a slight explanation comes at once to such steps as are given in the last two sentences quoted. The development much to be preferred *for all students* is that shown by the following excerpts from the 1912 "Syllabus of Mathematics" compiled by the committee on the teaching of mathematics to students of engineering under the direction of the Society for the Promotion of Engineering Education:[1]

"The ratio $\Delta y/\Delta x$ may be called the *average rate of change of the function during the interval from* $x = x_0$ *to* $x = x_0 + \Delta x$. (Geometrically, $\Delta y/\Delta x$ is the slope of the secant.) Now let Δx

[1] Copies can be obtained from the Secretary, Professor H. H. Norris, Ithaca, N. Y.

approach zero. . . . Then *the ratio $\Delta y/\Delta x$ will in general approach a definite limit, and this limit is called the actual rate of change at the point $x = x_0$* (geometrically, the limit of $\Delta y/\Delta x$ is the slope of the tangent).

"The *rate of change* of a function $y = f(x)$ at any point, or the *slope of the curve* at that point, is called the *derivative* of the function at that point, and is denoted by $f'(x)$, ... The value that Δy *would have* if the curve coincided with its tangent is called the *differential* of y. . . . The use of differentials gives us a *new notation for the derivative, $f'(x) = dy/dx$.*"

It should be said that in the last statement quoted it is implied, but was unfortunately not stated clearly, that the right-hand member is the quotient of two differentials.

To return to the text under review, the first chapter—a general survey of differential and integral calculus—is ably, because concretely, developed, presenting successively the notions of rate of change, formal derivative, differential, functions which have the same derivative, work required to stretch a spring, area under a parabola, differential equations, the definitions of indefinite integral and definite integral, with the symbol for the last. Inasmuch as about fifty problems are listed in this introductory chapter, it will be seen that the basic principles of both branches of the calculus need to be fairly mastered at the outset of the course. It would be a fair question to raise whether it is not crowding the student beyond his ability to give him all these ideas in one chapter, when, as we all know, the single abstract notion of differentiation is of such difficulty to many in our classes that it is only slowly grasped—*actually grasped* and handled as a working tool—by the beginner. If on the other hand it is not meant that these new notions should all be mastered at once, why present them all in one chapter? A weakness in Art. 7 may be noted here. Where the *temperature gradient* is defined, the argument is made that the temperature T along a rod must vary *continuously,* because "otherwise the temperature gradient of T at a point would be unthinkable." Only if the students were already acquainted with the idea of gradient, would this argument be convincing.

Chapter II develops in the usual fashion the standard formulas for differentiation. These are referred to in the next chapter, on "Integration," as rules for integration also. In addition, applications of these are made to velocity and acceleration, harmonic motion, curvature, and, as examples of geometric differentiation, to the acceleration of a particle moving in a circular orbit, and to the inward force per unit of length of a barrel hoop, with a commendable emphasis throughout on the proper units in which certain distances, velocities, accelerations, etc., should be expressed. See, for example, problem 1, page 36, and problem 10, page 58.

Still further evidence of the concrete character of the text is given by the very novel introduction in Chapter III of the planimeter to explain the idea of integration, as well as by reference to the cyclometer, electricity meter, and car "speed-time" diagram. Indeed, so completely is the emphasis upon the practical side that the whole technique of integration is disposed of in four pages, by referring to integration by inspection, by the aid of four of the simplest of the usual rules for integration, and, beyond this, by the use of the fairly complete

integral tables given at the end of the book. Of course, practice in these is given through a classified collection of some sixty-nine problems. In order to present an unbroken discussion of the principles, the problems for the second and third chapters are relegated to an appendix.

Chapter IV treats of partial differentiation and integration. Partial derivatives are presented by a clever use of the device of a hill built upon the xy-plane, this device being used consistently throughout the explanation of "gradient in two dimensions" and volumes and surfaces as double integrations.

Chapter V, on "Miscellaneous Applications," combines the topics of probable errors of derived results (applying partial derivatives), maximum and minimum values, problem of the bent beam, average value of a function, center of gravity and of pressure, and moment of inertia. The last includes the proof of the principle of parallel axes, but supplies only two problems to which this may be applied; the abbreviated measure of practice problems accorded to this and other important principles is evidently to be explained by the unusually extensive range of subjects transferrred to this course from the second and even third course in calculus, as traditionally arranged.

In consonance with the general method adopted for the book, Chapter VI gives a "plausible" proof of Maclaurin's theorem and from this derives Taylor's theorem, but evinces a desire to make no distinction between the two. The reviewer cannot but regard this as unwise, the former being universally used as a separate form, even though it be a special case of the latter. This chapter presents also Demoivre's theorem, hyperbolic functions, and Euler's equations which express $\sin x$ and $\cos x$ in exponential form. Except for illustrating the utility of these equations for the electrical engineer, the integration of $\sin mx \cos nx$ might better be carried out by means of the substitution:

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B).$$

Chapter VII describes certain ordinary differential equations, presents several well chosen illustrations from physics of the "principle of superposition," represented mathematically in the solutions of linear differential equations with constant coefficients; and, under the figure of such familiar examples as the starting and stopping of a boat and the undamped and damped oscillations of a weighted coiled spring, points to the fundamental problems of the theory of alternating currents.

Chapter VIII treats skilfully and clearly the partial differential equation of wave motion, applying it by the method of infinitesimals to the vibrations of a stretched string, and leading of course to an outline treatment of Fourier's sine and cosine series, the proofs as to the convergence and equivalence of the series being waived. That the gist of Fourier's series should be condensed into eight brief pages and in a manner that would seem to appeal readily to the *practical* thinker, is truly noteworthy; it is a happy example of "the new mathematics."

The final Chapter IX startles the reader still more with as simple and sensible a treatment of vector analysis as the writer remembers ever to have seen. True,

much that is in the chapter must be mastered through the instructor's exposition, but we must make proper recognition when we meet with a treatment of vectors that continually keeps before the student the knowledge that vectors *can be used* and that makes the theorem that the curl of a gradient is zero appear as a matter of common sense (even to a pure mathematician). This is done by means of a well-explained example of a curl. The chapter closes with suggestive, rather than complete, proofs of Stokes's theorem and the formulas for small displacements, with incidental discussion of vector fields and potential.

An appendix provides a careful but broad selection and description of texts to be recommended for the student's further study in mathematics and mathematical physics.

William DeW. Cairns.

PROBLEMS AND SOLUTIONS.

B. F. Finkel, Chairman of the Committee.

PROBLEMS FOR SOLUTION.

ALGEBRA.

398. Proposed by R. D. CARMICHAEL, Indiana University.

In the equation $x^3 + \alpha x + \beta = 0$, α is an integer divisible by p^2 and β is an integer divisible by p, p being a prime number. Prove that β is divisible by p^3 if the equation is reducible.

399. Proposed by W. H. BUSSEY, University of Minnesota.

A borrows from B \$1,500 and pays back \$34 a month for 63 months. If the last payment closes the account, what rate of interest has A been paying?

400. Proposed by C. N. SCHMALL, New York City.

Sum the series

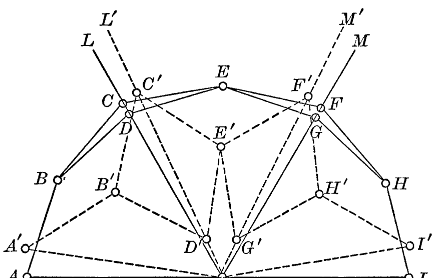
$$1 + 2x + 3x^2 + 4x^3 + \cdots$$

(Bromwich, *Infinite Series*, p. 129, ex. 1.)

GEOMETRY.

427. Proposed by F. CAJORI, Colorado College.

In S. Gross's linkage for trisection of angles, shown in the figure (KL' and KM' being the



trisectors of $A'KI'$), C is fixed on KL, also F on KM; at starting, C and D coincide, also F and G;

D slides on KL, G slides simultaneously on KM; if E moves along a perpendicular to AI erected at K, find the loci of B and D.[1]

428. Proposed by R. D. CARMICHAEL, Indiana University.

On a given chord of a circle as a base construct a right triangle with vertex outside of the circle such that its hypotenuse shall be bisected by its point of intersection with the circle. Are ruler and compasses sufficient to construct a triangle whose hypotenuse shall thus be divided in any ratio whatever?

429. Proposed by JOHN A. BIGBEE, Little Rock, Ark.

In the trihedral angle V-ABC, the face angle AVB is bisected by the straight line VD. Is it true that the angle DVC is less than, equal to, or greater than, half the sum of the angles AVC and BVC, according as $\angle CVD$ is less than, equal to, or greater than 90°?

430. Proposed by DANIEL KRETH, Wellman, Iowa.

The distance between A and B is always a feet. A travels along a straight path at the rate of v_1 miles per hour, and B starts at the same time in the path behind A and travels in a curve at the rate of v_2 miles per hour. How far will B travel to reach the path in front of A, and how far to reach the path again behind A?

CALCULUS.

350. Proposed by R. P. BAKER, University of Iowa.

Find a general formula for $\dfrac{d^ny}{dx^n}$ in terms of $\dfrac{d^ky}{dt^k}$ and $\dfrac{d^kx}{dt^k}$.

351. Proposed by C. N. SCHMALL, New York City.

In the ellipse $(x^2/a^2)+(y^2/b^2)=1$, are given the eccentricity e and the angle ϕ which the normal at any point P (on the curve) makes with the major axis. If R is the radius of curvature at P, prove that

$$R=\frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}}.$$

352. Proposed by RICHARD P. LOCHNER, Philadelphia, Pa.

At point P there are n foxes. At Q, a rods south of P, there is a dog. The dog and the foxes are freed at the same instant and run at uniform speeds. Some of the foxes run east, some north, some west and some south. The dog runs first toward the foxes that ran east and always points toward them. He captures one of them and then instantly pursues the pack that ran north. In like manner, when he has captured one of them, he pursues those that ran west, then those that ran south, and then begins over again by pursuing the ones running east. If r is the ratio of the dog's speed to that of a fox, what is the total length of the n curves of pursuit?

(Generalization of a problem published in 1859 in the *Mathematical Monthly*.)

MECHANICS.

284. Proposed by C. N. SCHMALL, New York, N. Y.

A cylindrical vessel standing upright on a horizontal plane is kept constantly full of water by an automatic device. Determine at what height in its side a small orifice should be made, so that the water may spout through it to the greatest horizontal distance on the plane. What difference in the result when the cylinder stands on a shelf of known height above the plane?

Note. This problem may have an application in the case of automatic sprinklers.

285. Proposed by RICHARD P. LOCHNER, Philadelphia, Pa.

A ladder 25 ft. long, weighing 120 lb., leans against a vertical wall. Its foot is prevented from slipping on the plane by a peg driven into the ground 7 ft. from the wall. If a man weighing 150 lb. is one-third the way up the ladder, what is the reaction on the peg, the ground, and the wall?

[1] Linkages for the trisection and multisection of angles are described in *Verhandl. des 3. Internationalen Mathematiker-Kongress* in Heidelberg, Leipzig, 1905, pp. 492–496; *Zeitschr. f. Math. u. Physik*, Vol. 49, 1903, pp. 342–347; *Mémoires de la société r. d. sciences de Liége*, 2. série, T. XX, 1898; *Ueber spezielle alg. und trans. Kurven*, von G. Loria (uebers. v. Schütte), Leipzig, 1902, *Sextrix-Kurven*, pp. 316–323.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

203. Proposed by R. D. CARMICHAEL, Indiana University.

Find solutions in integers of the equation $2x^2 + 1 = 3y^2$.

204. Proposed by E. T. BELL, New York City.

Show that a necessary and sufficient condition that $6n + 1$ be a prime number is that no one of the quantities $(3n - r)/(2r + 1)$ for $r = 1, 2, 3, \cdots n - 1$ be an integer; similarly for $6n - 1$, the quantities being $(3n - r)/(2r - 1)$ for $r = 2, 3, 4, \cdots n$.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

389. Proposed by W. W. BEMAN, University of Michigan.

If $e^{e^x} = 1 + a_1x + a_2x^2 + a_3x^3 + \cdots$, prove that

$$na_n = \sum_{k=1}^{k=n} \frac{1}{(k-1)!} a_{n-k} \quad \text{or} \quad n!\, a_n = \sum_{k=1}^{k=n} \frac{(n-1)!}{(k-1)!} a_{n-k}.$$

Solution by A. M. Harding, University of Arkansas.

We shall assume that the series

$$e^{e^x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

may be differentiated term by term. We then obtain

$$e^x \cdot e^{e^x} = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + \cdots$$

or

$$\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots\right)\left(a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + \cdots\right)$$
$$= a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + \cdots.$$

Multiply the two series and equate coefficients of like powers of x. We then have

$$a_1 = a_0,$$

$$2a_2 = a_1 + \frac{a_0}{1!},$$

$$3a_3 = a_2 + \frac{a_1}{1!} + \frac{a_0}{2!},$$

$$4a_4 = a_3 + \frac{a_2}{1!} + \frac{a_1}{2!} + \frac{a_0}{3!},$$

$$\cdots = \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$na_n = a_{n-1} + \frac{a_{n-2}}{1!} + \frac{a_{n-3}}{2!} + \frac{a_{n-4}}{3!} + \cdots \frac{a_0}{(n-1)!}$$
$$= \sum_{k=1}^{k=n} \frac{1}{(k-1)!} a_{n-k}.$$

Solved similarly by A. L. McCarty and Elmer Schuyler.

390. Proposed by E. B. ESCOTT, University of Michigan.

Sum the series

$$\frac{1}{1}+\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{5}{16}+\frac{8}{32}+\cdots,$$

where each numerator is the sum of the two preceding and the denominators are in geometrical progression.

Solution by R. D. Carmichael, Indiana University.

The series may be written in the form

(1) $$S = 1 + \sum_{n=1}^{\infty} \frac{u_n}{2^n},$$

where

(2) $$u_{n+2} - u_{n+1} - u_n = 0, \qquad u_1 = 1, \qquad u_2 = 2.$$

We first solve the recurrence relation (2) without restriction of initial values. Substituting ρ^n for u_n, we find that (2) will be satisfied if

$$\rho^2 - \rho - 1 = 0;$$

that is, if ρ has either of the values

$$\rho_1 = \frac{1+\sqrt{5}}{2}, \qquad \rho_2 = \frac{1-\sqrt{5}}{2}.$$

The general solution of (2), without restriction of initial values, is therefore

$$u_n = \alpha\rho_1^n + \beta\rho_2^n,$$

where α and β are constants.

We must now determine α and β so that $u_1 = 1$ and $u_2 = 2$; that is, so that

$$\alpha\rho_1 + \beta\rho_2 = 1,$$

$$\alpha\rho_1^2 + \beta\rho_2^2 = 2.$$

Thus we have readily

$$\alpha = \tfrac{1}{10}(5+\sqrt{5}), \qquad \beta = \tfrac{1}{10}(5-\sqrt{5}).$$

Putting in (1) the value of u_n thus determined, we have

(3) $$S = 1 + \tfrac{1}{10}(5+\sqrt{5})S_1 + \tfrac{1}{10}(5-\sqrt{5})S_2,$$

where

$$S_1 = \sum_{n=1}^{\infty} \frac{(1+\sqrt{5})^n}{2^n \cdot 2^n}, \qquad S_2 = \sum_{n=1}^{\infty} \frac{(1-\sqrt{5})^n}{2^n \cdot 2^n}.$$

Summing these geometric series, we have

$$S_1 = 2+\sqrt{5}, \qquad S_2 = 2-\sqrt{5}.$$

Substituting in (3), we obtain

$$S = 4,$$

the result sought.

Also solved by H. A. Levy, A. M. Harding, J. A. Colson, Levi S. Shively, H. C. Feemster, Elmer Schuyler, and the Proposer.

391. Proposed by C. N. SCHMALL, New York, N. Y.

Show that the roots of the quadratic

$$ax^2 + 2bx + c = 0$$

are imaginary if a, b, c are in harmonic progression and have the same sign.

Solution by S. W. Reaves, University of Oklahoma.

Since a, b, c are in harmonic progression,

$$b = \frac{2ac}{a+c}.$$

The discriminant of the given quadratic then becomes

$$b^2 - ac = \left(\frac{2ac}{a+c}\right)^2 - ac = -\,ac\left(\frac{a-c}{a+c}\right)^2,$$

which is negative since a and c have the same sign. The discriminant being negative, the roots are imaginary.

Also solved by Walter C. Eells, Elmer Schuyler, F. M. Morgan, H. C. Feemster, J. A. Colson, and A. M. Harding.

Additional solutions of 387 were received from R. M. Mathews, Horace Olson, and G. Y. Sosnow, after the December issue had gone to press.

Note. We have solutions of all problems proposed during 1913 in this section up to the September number, except 385, which was published in February.

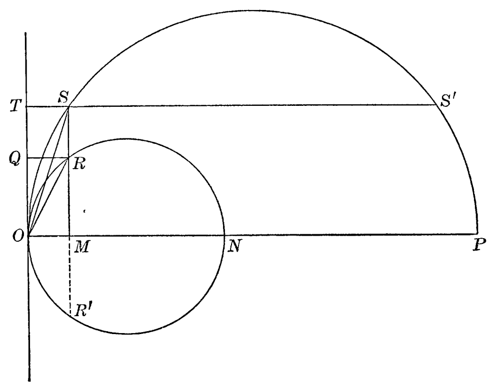
GEOMETRY.

419. Proposed by S. LEFSCHETZ, University of Nebraska.

Given a circle and a tangent to it. To find a point on its circumference such that the sum of its distances to the tangent and its point of contact shall be equal to a given length.

Solution by S. W. Reaves, University of Oklahoma.

Let ORN be the given circle, OT the given tangent, and O the point of contact. Denote the given length by l.



Construction. Draw the diameter ON and extend it beyond N to P making $NP = 2l$. On OP as diameter construct the semi-circle $OSS'P$. On the given tangent lay off $OT = l$. Through T draw the line TSS' parallel to OP to meet the semi-circumference first in S. Drop a perpendicular from S to OP meeting OP in M and meeting the given circle in R and R'. Then R (or R') is the required point.

Proof. Let $QR = OM = x$, $OR = y$, $OS = z$, $ON = 2r$. Then by construction $OP = 2(r + l)$ and $MS = l$.

By elementary geometry $OM : OR = OR : ON$.

Hence

$$y^2 = 2rx. \tag{1}$$

Similarly

$$OM : OS = OS : OP.$$

Hence,

$$z^2 = 2x(r + l), \tag{2}$$

or, from (1),

$$z^2 = y^2 + 2lx. \tag{3}$$

But $z^2 = x^2 + l^2$, and therefore (3) becomes

$$y = \pm (l - x). \tag{4}$$

Since $y > 0$ and $l > x$, the lower sign is inadmissible. Hence $x + y = l$, or

$$QR + OR = l.$$

A solution exists when and only when $l \leqq 4r$, and there is not more than one solution on the same side of the diameter through the point of contact.

Solutions were also received from WALTER C. EELLS, H. C. FEEMSTER, C. N. SCHMALL, ELMER SCHUYLER, and A. H. HOLMES.

420. Proposed by C. N. SCHMALL, New York City.

Four spheres are described so that each touches a face of a given triangular pyramid and the other three faces produced. If the radii of the escribed spheres be r_1, r_2, r_3, r_4, and r be the radius of the inscribed sphere, show that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{r}.$$

SOLUTION BY HORACE OLSON, Chicago, Illinois.

Let $ABCD$ be a given tetrahedron.

Let the faces ABC, ABD, ACD, and BCD be represented by small letters corresponding to the opposite vertices of the tetrahedron, *i. e.*, by d, c, b, and a respectively.

Let V represent the volume of the tetrahedron $ABCD$.

Let O be the center, and r the radius, of the inscribed sphere.

Let O_1, O_2, O_3, and O_4 be the centers, and r_1, r_2, r_3, and r_4 the radii, of the escribed spheres opposite to A, B, C, and D, respectively.

Connect points O and O_3 with the 4 vertices of the tetrahedron.

Then,

$$V = AOBC + OBCD + OACD + OABD = \frac{ar}{3} + \frac{br}{3} + \frac{cr}{3} + \frac{dr}{3} = \frac{r}{3}(a + b + c + d),$$

and

$$r = \frac{3V}{a+b+c+d}.$$

Also

$$V = AO_3BC + O_3BCD + O_3ACD - O_3ABD = \frac{r_3}{3}(a+b-c+d)$$

and

$$r_3 = \frac{3V}{a+b-c+d}.$$

Similarly,

$$r_1 = \frac{3V}{(-a+b+c+d)}, \quad r_2 = \frac{3V}{(a-b+c+d)}, \quad \text{and} \quad r_4 = \frac{3V}{(a+b+c-d)}.$$

Therefore

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4} = \frac{(-a+b+c+d)+(a-b+c+d)+(a+b-c+d)+(a+b+c-d)}{3V}$$

$$= \frac{2a+2b+2c+2d}{3V} = \frac{2}{r}.$$

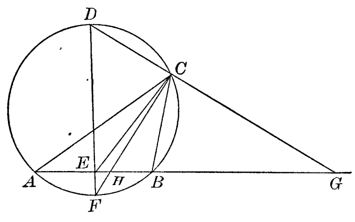
Also solved by A. M. HARDING and ELMER SCHUYLER.

422. Proposed by J. SCHEFFER, Hagerstown, Md.

Construct strictly by use of elementary plane geometry, a triangle having given the product of two sides, the median to the third side, and the difference of the angles to the third side.

SOLUTION BY F. M. MORGAN, Dartmouth College.

Suppose ABC is the required triangle circumscribed by a circle. Draw the internal and external bisectors of $\angle C$ meeting the circle in points F and D. Draw FD.



Since $\angle DCF$ is a right angle, FD is a diameter and bisects AB. Produce DC to meet AB in G. Since $\angle FEG = \angle FCG = 90°$, a circle will pass through points F, E, C, G. Therefore, $\angle EFH = \angle EGC = \frac{1}{2}$ of the difference of the base angles of triangle ABC. This angle is given and will be called α.

Now $\angle EHC = 90° + \alpha$ and is therefore known. Moreover, $CH \cdot CF = CA \cdot CB$ which is a given product.

Construction. Take line CE equal to the given median. Upon it describe the segments of two circles, one to contain an angle α, and the other an angle $90°+\alpha$. Draw line through C cutting these circles so that $CH \cdot CF = CA \cdot CB$. To do

this produce CE to I so that $CE \cdot CI = CA \cdot CB$ and construct an angle at I equal to $90^\circ + \alpha$. Then F is the point where the side of this angle meets the segment containing $\angle \alpha$; for E, H, F, I are on a circle and $CE \cdot CI = CH \cdot CF$.

Draw FE and let it meet the perpendicular to FC at C in point D. Pass a circle through D, C, F and where this circle meets EH extended will be points A and B. The triangle ABC is then the required triangle.

Also solved by A. H. HOLMES, DAVID F. KELLEY, ELMER SCHUYLER, and A. M. HARDING.

A solution of 416 was received from S. W. REAVES too late for notice in the November, 1913, issue.

No solution has been received for 421.

CALCULUS.

A solution of 334 has been received to which the author neglected to sign his name. Two solutions of 327, which was reprinted in the February, 1913, issue, p. 68, have been received. Also solutions are in hand for 336, 339, 341, and 343. All of these will appear in the next issue. Meanwhile we hope to hear from 337, 338, 340, 342, and 344, which include all published up to November, 1913.

MECHANICS.

No solutions have been received for the following problems in mechanics: 246 published in October, 1911; 266 in December, 1911; 268 in January and 269 in April, 1912, and none of those published in 1913, namely 271–283.

Solutions of these problems are desired.

Please remember that problems 276–279 are incorrectly numbered 271–274. See page 258 of the October, 1913, issue.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

190. Proposed by H. C. FEEMSTER.

Show that, if n is a prime number and r an integer less than n, then

$$(r-1)!\,(n-r)! + (-1)^{r-1} = M \cdot n,$$

where M is an integer.

SOLUTION BY THOMAS E. MASON, Indiana University.

From the Wilson theorem we have

$$(n-1)! + 1 \equiv 0 \mod n. \tag{1}$$

We have also,

$$\begin{aligned}(n-1)(n-2)\cdots(n-r+1) &\equiv (-1)(-2)\cdots(-r+1) \mod n.\\ &\equiv (-1)^{r-1}(r-1)! \mod n. \end{aligned} \tag{2}$$

Making substitution from (2) in (1) we get

$$(-1)^{r-1}(r-1)!(n-r)! + 1 \equiv 0 \mod n,$$

or

$$(r-1)!(n-r)! + (-1)^{r-1} \equiv 0 \mod n,$$

which proves the theorem.

Also solved by B. F. YANNEY, L. C. MATHEWSON, and the PROPOSER.

Note. No solutions have been received for problems proposed in 1913 under this heading, except for 188 and 190. Solutions for the remaining ones are desired. Please remember that 191 to 196 are incorrectly numbered 187–192. See page 258 of the October, 1913, issue.

MISCELLANEOUS QUESTIONS.

EDITED BY R. D. CARMICHAEL.

The new department of "Miscellaneous Questions" has met with a pleasing reception among our readers; and we desire to express our appreciation of the interest which has already been manifested through the contributions of questions and replies. We hope to have the coöperation of a considerable number of persons in realizing the double purpose announced in the issue of November, 1913. One of our correspondents, who is a physicist, has written concerning one of these purposes; and we desire to call attention to his remarks in order that we may invite our readers to assist us in realizing the advantages which are mentioned:

"I believe that your department of miscellaneous questions could render valuable service in bringing together the mathematicians and the physicists. Generally the physicist does not know his mathematics sufficiently well to enable him to handle original problems, while the mathematician cannot keep in touch with the developments in physics; but it ought to be possible to work out a plan so that the combined skill of the mathematician and the physicist could be utilized in the solution of the many physical problems. It strikes me that your department of miscellaneous questions could be developed so as to cover this field, and I believe that it would be very useful."

In this issue we have the first replies to our questions. This is the time to say that a question is not disposed of because we have published some answers to it. We wish to have it understood clearly that, although it is desirable to have early replies to our questions, it is yet never too late to send in other replies. In fact, we believe that it will often happen that one set of answers to a given question will provoke others; and that the same question may often be discussed through several issues. Question 2 is one which is probably of this kind.

Again, we shall often print partial answers to a given question in order to call forth a further or more complete answer. This we are doing in the present issue in the case of question 1.

QUESTION.

5. In what ways may mathematics contribute most to the culture of the individual? What is being done and what may be done to advantage in the matter of developing courses in culture mathematics?

REPLIES.

1. In connection with the investigation of a problem in physics Mr. Louis Cohen, Washington, D. C., requires to have expressed in the form of a Fourier's series

$$y = \Sigma A_n \cos nt + \Sigma \beta_n \sin nt \tag{1}$$

the general solution of the differential equation

$$\frac{dy}{dt} + (\alpha + \beta \cos t)y = \rho \cos t, \tag{2}$$

where α, β and ρ are constants. Is the solution of this problem contained directly in matter which has already been published? If not, how may one proceed to the determination of the constants A_n and B_n so that (1) shall be the general solution of (2)?

I. REMARKS BY C. N. HASKINS, Lebanon, N. H.

With reference to the equation in the problem it may be noted that since the general solution is

$$y = \rho(ce^{-(\alpha t+\beta \sin t)} + e^{-(\alpha t+\beta \sin t)} \int e^{\alpha t+\beta \sin t} \cos t dt,$$

a solution in the form

$$y = \rho(ce^{-\alpha t}F_1(t) + F_1(t)F_2(t)),$$

where $F_1(t)$ and $F_2(t)$ are Fourier's series, can be found by noting that the coefficients of the Fourier expansion of $e^{\beta \sin t}$ are simple functions of the quantities $I_n(\beta)$, where

$$I_n(\beta) = i^{-n}J_n(i\beta),$$

the Bessel function of order n with pure imaginary argument. Having found the coefficients in $F_1(t)$ and $F_2(t)$ those in their product may be computed, so that the solution takes the form

$$y = \rho(ce^{-\alpha t}F_1(t) + F(t)),$$

where $F(t)$ and $F_1(t)$ are Fourier's series. This is not the form of solution demanded; but it would appear to be a preferable form for some purposes at least, since the strictly periodic terms are separated from those which are affected with the damping factor $e^{-\alpha t}$ and the arbitrary constant c.

II. NOTE BY THE EDITOR.—The above remarks suggest a means of finding a solution in a form different from that required in the problem. It is given because it seems to be of interest in itself and because further it may be of value in securing the desired solution. We shall be glad to have further communications relative to this problem. Concerning its importance note the following remark by the proposer: "The problem itself is of great importance; a satisfactory solution of this differential equation would lead to a very interesting development of a group of problems in electrical theory."

2. It is clear, on the most casual observation, that the average college curriculum in mathematics is unbalanced in two respects: (1) algebra predominates over geometry, (2) analytic geometry predominates over our synthetic geometry. This reacts in two undesirable ways: (1) to deprive the college student of a rich and interesting field of study, (2) to give a very one-sided training for prospective teachers of high school mathematics. What can be done to remedy this situation?

I. REMARKS BY D. N. LEHMER, University of California.

In reading the paper on "Synthetic Projective Geometry" in the November issue of the MONTHLY, I was a little surprised to see that after all our teaching of tha subject in the University of California for at least ten years, with some little

research and very great enjoyment on the part of the students, the work should not have attracted enough notice to be mentioned. We are giving the course to sophomores and even to some freshmen, with and without the calculus, and we think that it has been the salvation of many students. There comes a time in the life of every mathematical student when the continual grind of analysis becomes unbearable. I have felt it myself, and I do not doubt that others have felt it also. Then is the time to give him something really beautiful for a relief. The constant insistence on rigorous proofs and existence theorems does not encourage one in the delusion that the subject of mathematics is one long vista of eternal joy.

I cannot find any particular reason why the calculus should be considered as a "prerequisite" for the subject. If it is supposed that no one can appreciate it without several years of analysis to prepare the way, it only indicates that it has never been tried. I have had freshmen do fine work in the subject and be so filled with the beauty of it that they immediately changed all their plans and went in for mathematics. I have a course in "culture mathematics" and a taste of this study often proves to be the salvation of the course. I think I do not exaggerate when I say that it is the most popular and most thoroughly enjoyed of all the courses given here in mathematics. It is required of all candidates for teachers' certificates in mathematics.

It is gratifying to know that others are waking up to the possibilities of this subject. If, as Mr. Bussey says in his article, the subject is out of the highways of workers in research (and I am not ready to grant that) there is no question of its real value as a relief from the dreary grind of analysis.

II. Remarks by C. E. Stromquist, University of Wyoming.

I have just read the article in the November number of the Monthly on "Synthetic Projective Geometry as an Undergraduate Study." I was very much interested, as I have a class in projective geometry this semester and many of the questions taken up regarding a first course in projective geometry have presented themselves to me. I am not certain whether I agree that a student who is preparing to do graduate work in mathematics "can afford to wait and take projective geometry when he is a graduate." My suggestion would be that he take an introductory course while an undergraduate student in order to be prepared for a more advanced graduate course. But our problem out here in Wyoming is not so much to arrange courses for prospective graduate students as to solve the problem of how to best fit those (undergraduate) students in our mathematical courses who are preparing to teach mathematics in our high schoools. I heartily agree with what Professor Bussey has to say regarding these students and think that his paper is a very timely one.

For a text-book I am using Hatton's "Principles of Projective Geometry" (published by the Cambridge University Press, 1913). In my judgment it is a very satisfactory text for a first course. It covers more ground than one could expect to cover in a half-year course, with three hours a week; but this is not such a serious objection. It contains several sets of problems. There are about 250 of these problems, not including several easier problems placed in the body of the ext.

III. Remarks by T. G. Rodgers, Normal School of New Mexico.

On graduating from a state university several years ago I began teaching high school mathematics. I found that aside from the larger vision gained in undergraduate study in the university I was no better prepared to teach geometry than on completing the high school course. Observation soon convinced me that others too were teaching merely the geometry which they had learned in the high school. An introduction to projective geometry in subsequent graduate work was of benefit but left me still feeling inadequately prepared to teach elementary geometry as a living subject.

My continued search for better methods was finally rewarded through my forming an acquaintance with the works of Rouche and Comberousse, Henrici and Treutlein, and Hadamard. The study of these works has enriched my teaching of geometry to such an extent that I look back with regret to the years in which I tried to teach the subject with only such preparation as the colleges give and with the feeling that college curricula are badly balanced in respect to their preparation of teachers for high school mathematics.

It appears to me that at least one year's work should be given to the study of synthetic geometry, the first half being a lecture course based on works like those mentioned above (especially Hadamard) and the second half being the kind of course in projective geometry outlined by Professor Bussey in the November issue of the Monthly.

NOTES AND NEWS.

Under the direction of Florian Cajori.

Please note that the date on your address label for this issue is changed if your renewal subscription was received before January 10, 1914.

The October number of the *Monist* contains an article by Professor Bertrand Russell on "The Philosophical Importance of Mathematical Logic."

The November number of *School Science and Mathematics* contains an article entitled "A Thread of Mathematical History and Some Lessons," by Professor R. D. Carmichael, of the University of Indiana.

The November number of the *School Review* contains an interesting paper entitled "The Cumulative Examination in Mathematics," by Mr. H. E. Webb, of Newark, N. J. He suggests a series of cumulative examinations extending over the three or four years of high school mathematics.

Dr. Heinrich Burkhardt's "Theory of Functions of a Complex Variable" has been translated into English by Professor S. E. Rasor of the Ohio State University, and published by D. C. Heath & Company.

Spon & Chamberlain have published a book on "The Gyroscope: its Theory and Practise" by Dr. F. J. B. Cordeiro, of the U. S. Navy. While abroad the author made a special study of the gyroscopic compass.

ALBERT W. WHITNEY, formerly associate professor of insurance and mathematics in the University of California, has become assistant actuary in the insurance department of the state of New York.

FRIEDRICH DINGELDEY's *Sammlung von Aufgaben zur Anwendung der Differential- und Integralrechnung* (B. G. Teubner) contains a collection of problems on the application of the calculus to geometry and also to physics, chemistry and technology.

The Southwestern Section of the American Mathematical Society met at the University of Missouri, Columbia, Mo., on November 29, 1913. There were about twenty members present and twelve papers were presented at the two sessions. Those who read papers were Professors E. R. Hedrick, Louis Ingold, W. C. Brenke, W. D. Westfall, O. D. Kellogg, Drs. E. L. Dodd, S. Lefschetz, A. J. Kempner, H. Blumberg, Mr. A. R. Schweitzer and Miss E. A. Weeks.

The annual meeting of the American Mathematical Society was held in New York on December 30 and 31, 1913.

The American Association for the Advancement of Science held its annual convocation at Atlanta, Ga., during the week beginning December 29, 1913. The retiring president of Section A, Mathematics and Astronomy, was Professor E. B. Van Vleck, of the University of Wisconsin, who gave his address at this meeting on the "Influence of Fourier's Series on Mathematics." The new president is Professor Frank Schlesinger, of Allegheny Observatory, and the secretary is Professor F. R. Moulton, of the University of Chicago.

The Northeastern Ohio Teachers' Association met at Cleveland, October 24 and 25. At the departmental meeting on mathematics papers were read as follows: "How to make the transition from arithmetic to algebra," "What should be included and what excluded in first-year algebra," "Some specific suggestions for teaching the pupil how to attack originals in geometry."

Professor CARLO BOURLET of L'École Nationale, Paris, died August 12, as the result of injuries received in an accident. Aside from notable work as a lecturer in higher mathematics, he has contributed in an important way to the teaching of elementary subjects, chiefly through texts in both algebra and geometry.

The American Mathematical Society has accepted the invitation of Brown University, extended through the committee on the celebration of her 150th anniversary, to hold the summer meeting at Providence in September, 1914.

The first part of Tome II, Volume 6, of the *Encyclopédie des Sciences Mathématiques*, was published on September 15, 1913. This fascicule of 128 pages is devoted to calculus of variations, and was edited by M. Lecat. It it interesting to observe that a considerable part of the space is devoted to a consideration of the results obtained by American mathematicians. Foremost among these are the contributions by Professor OSCAR BOLZA, who returned to Germany a few years ago after having done very much to advance American mathematics, both

by his own work during his twenty years of residence here, and also by the inspiration which he gave to a large number of young investigators.

The *Mathematical Gazette* for July and October, 1913, contains an article by Sydney Lupton on "The radix method" of computing logarithms, which possesses certain advantages when logarithms and antilogarithms are required to many places and, in default of tables, must be calculated. The method requires numbers of the form $(1 + r/10^n)$, with their logarithms to one or two places beyond those actually to be used. The method goes back to Briggs, 1624, and was simplified by Robert Flower, 1771, and further modified in 1786, by George Atwood of "machine" fame. The article contains many historical notes on the computation of logarithms. Several nineteenth century writers rediscovered the old method of Briggs and Flower.

During the Annual High School Conference held at the University of Illinois, November 20, 21, and 22, 1913, the following papers were presented before the mathematics section: "The use of the Courtis tests," by Professor L. D. Coffman, University of Illinois; "Statistical method in educational studies," by Professor A. C. Lunn, University of Chicago; "Bibliography on methods of grading," by Mr. H. C. Zeis, University of Illinois. There was also held a "Round table discussion on methods of grading." According to the program, "All superintendents, principals and high school teachers, as well as teachers in academies, normal schools, colleges and universities are invited to be present and participate in the discussion."

The Central Association of Science and Mathematics Teachers held its thirteenth annual meeting at Des Moines, Iowa, on November 28 and 29, 1913. Professor Florian Cajori was one of the leading speakers at the opening general session and also at one of the sessions of the mathematics section. At the former he spoke on "Science and mathematics in vocational schools; a retrospect," and at the latter on "Means of measuring mathematical abilities." Other numbers on the two programs of the mathematics section were: "Report of a committee on vocational mathematics," by R. L. Short of Cleveland, Ohio; "Report of a committee on examinations and results" by C. E. Comstock of Peoria, Ill., and a paper on "The traditional examination in mathematics" by Jane V. Pollock of Kenilworth, Ill. The report on vocational mathematics was discussed by Professor G. A. Smith, Iowa City, W. Lee Jordan, Des Moines, and W. G. Swartz, Gary, Ind. The association is one of the largest and most active in the secondary field. The mathematics section has put forth at various times some important reports which have had wide influence; in particular, its report on algebra some years ago had a circulation of over ten thousand copies and has without doubt had a marked influence on most of the texts on elementary algebra published since that time. This association was organized in Chicago in 1901 and most of its meetings have been held in Chicago.

The Chicago Section of the American Mathematical Society held its thirty-second regular meeting at the University of Chicago on December 26 and 27,

1913. There were thirty-five papers on the program and 56 members were in attendance. The Chicago section was organized in 1897 and has steadily grown in importance both as to the number of members of the Society attending its meetings, and as to the number of scientific papers presented. The Society itself was organized in New York in 1894, and the regular meetings of the Society are usually held in New York (see a paper on "Western Meetings of Mathematicians" in the April, 1913, issue of the MONTHLY). The Council, however, has recently voted to designate the meetings of the Chicago section as "meetings of the Society at Chicago," and the meeting just held was the first under this designation. In honor of the occasion the University of Chicago tendered a complimentary dinner to the members present.

The *Mathematical Gazette* for October, 1913, contains an exceedingly interesting account, by Dr. C. G. KNOTT, of the Edinburgh mathematical colloquium, held last August, at which three courses of lectures were given, respectively, by an Englishman, E. P. WHITTAKER; an Irishman, A. W. CONWAY; and a Scotchman, D. M. Y. SOMMERVILLE. Conway discoursed on the theory of relativity; Whittaker explained in his mathematical laboratory practical harmonic analysis and periodogram analysis; Sommerville expounded non-euclidean geometry. We quote the following:

"Professor Whittaker chose for his working data the light periods of two variable stars, the one to illustrate the periodigram method of discovering unknown periods, the other to illustrate the analysis into harmonic components of a given periodic variation. The theory of the Fourier analysis was incidentally given; and the last lecture finished with an account of Mäder's Harmonic Analyser."

"This hour of practical work, combined with demonstrations, involving only the familiar circular functions, gave the necessary balance to the weird imaginings of the other two courses. Without it to bring us back to the obvious world of apparent realities we should have been floundering hopelessly in the Absolute or in Minkowski's *Welt*. After we had been taught that velocities did not compound according to the parallelogram law, it was a positive delight to find that the Fourier series remained ordinarily additive; and with this in possession we had no great difficulty in apprehending the possibility of a space devoid of parallel lines. . . ."

"We learned many things. We were told that even if the theory of relativity were not true it had taught us truths. The tendency of modern physical theory was in the direction of still further atomising the atom; yet it was necessary in geometry to have an assumption of continuity, so that all possible numbers might be brought into correspondence with an infinitude of points on a finite line. The dictum of the logician that we cannot define by means of a negation seemed to have no terror to the modern geometer with his glib talk on non-Euclidean, non-Pascalian, non-Desarguesian, and even non-Archimedean."

In the spirit of the editorial statement on the first page of this sisue, the action recently taken by the mathematics teachers of California, as indicated in the following communication, is most significant. The editors of the MONTHLY appreciate the compliment and recognize the responsibility which this action implies. We congratulate the teachers of California upon being pioneers in taking such a bold, forward step.

The communication, under date of January 2, 1914, is from Professor Henry W. Stager, of Fresno Junior College, chairman of the mathematics section of the California High School Teachers' Association, and is addressed to the teachers of mathematics in the secondary schools of California. It reads as follows:

"At the summer meeting of the Mathematics Section of the California High School Teachers' Association, a Committee, composed of Professor D. N. Lehmer, Miss S. Gilmore, and Professor G. A. Miller, was named to consider ways and means of broadening the scholarship of secondary school teachers. The Committee recommended the reading of the book, *Mathematical Monographs,* by J. W. A. Young in collaboration with other mathematicians, and further stated: 'In connection with this book, or even instead of it, the reading of some elementary but strictly first class journal is also recommended. Where it is possible for a number of teachers to meet frequently for the discussion of problems and short articles this plan has much to commend it. For this purpose the Committee would recommend the AMERICAN MATHEMATICAL MONTHLY.'

"The report has been adopted unanimously. The movement for a higher standard of efficiency is one of the strongest attempts to increase the quality of the teaching of mathematics ever made in California. It began with the teachers themselves and only needs your coöperation to make it a success. I commend the report of the Committee to your most earnest consideration.

"At this time I wish especially to call your attention to the reading of a first class journal. I have gone over with care the first volume of THE AMERICAN MATHEMATICAL MONTHLY under the new organization and find it well fitted for this purpose. I am personally acquainted with many members of the Publication Committee and with all the members of the Editorial Committee. The men are in the forefront of the present movement toward a higher standard for the teaching of mathematics in the United States. The MONTHLY represents the best in its field. I commend it to you, feeling sure that it will prove of very great benefit to you, will help to increase your efficiency, and will be a continual source of inspiration to you.

Yours, for the better teaching of mathematics,

HENRY W. STAGER.

THE

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THE ALGEBRA OF ABU KAMIL.

By L. C. KARPINSKI, University of Michigan.

Popularity of Algebra Among the Arabs. The great interest in the study of algebra among the Arabs of the Middle Ages is attested by the numerous works written upon this subject by Mohammedan mathematicians. The period of this popularity extends from the ninth to the fifteenth century. The students of algebra included poets, philosophers, and kings. The popular Omar Khayyam was too excellent a mathematician and too fine a poet to woo the muse of algebra in verse. But his treatise in prose on algebra, edited with French translation by Franz Woepcke, is as good a title to fame as his verse. In the library of the Escurial is preserved a *poem* written six hundred years ago by a native of Granada, Mohammed al-Qasim, and treating of algebra. Needless to say the content does not compare with the prose of Omar Khayyam. The King of Saragossa Jusuf al-Mutamin (reigned 1081–1085) was a devoted student of the mathematical sciences. The title of one of his works would seem to indicate its algebraical nature. Even in the study of law a knowledge of algebra seems to have been necessary, for various questions of inheritance were treated by this science. On the whole, however, in that happy day the question of practical applications of mathematics did not loom so large upon the horizon of the Arabic scientist. That fortunate individual felt entirely free to pursue this subject as one of the natural and inevitable activities of the human intellect.

The First Great Arabic Writer on Algebra. The first systematic treatise on algebra was the product of an Arabic writer, Mohammed ibn Musa, al-Khowarizmi (c. 825 A. D.). Commentaries were written upon this work by Arabic authors and two different Latin translations were made of it. One of these translations was published by Libri in his *Histoire des sciences mathématiques en Italie,* Vol. I, 253-297, Paris, 1838. This work is out of print. While the authorship of this version is uncertain the composition may safely be placed in the twelfth century, as several writers of that century appear to have studied this work and use the terminology of this translation. Manuscript copies are somewhat common in

European libraries. French, German, and Italian writers of the fifteenth and sixteenth centuries drew inspiration, and copy, from these pages. The other Latin translation was effected in 1144 A. D. (1183 of the Spanish era) by an Englishman, Robert of Chester, the first translator of the Koran. This version is found, so far as known, only in three manuscripts.[1] The text with English translation and notes is being published in Vol. XI, *University of Michigan Studies, Humanistic Series*. The Arabic text with English translation was published by F. Rosen, London, 1831, but unfortunately this last is out of print. Adequate treatment of this initial work in the science of algebra is found in Cantor's *Vorlesungen über Geschichte der Mathematik*, volume I.

The Second Great Arabic Writer on Algebra. The name and fame of Al-Khowarizmi is known to most students of mathematics. But fate has not dealt so kindly with the second, in point of time, of the great Arabic writers in this field. Abu Kamil Shoja' ben Aslam[2] wrote in the late ninth or early tenth century a much more extensive treatise on algebra than that of Al-Khowarizmi. Two commentaries of the tenth century have been noted by Arabic historians. In addition to these a commentary in Arabic was prepared by a Spanish Arab, Al-Khoreshi, and a Spanish translation by an unknown Christian Spaniard.[3] This was followed by a Hebrew translation, which is preserved in manuscript, by Mordechai Finzi (c. 1473) of Mantua. Not alone by these translations and commentaries was the influence of Abu Kamil exerted upon later writers in this science; for two prominent algebraists made large use of his treatise without, however, mentioning his name. These two men are Al-Karkhi (died c. 1029), whose treatise entitled *Al-fakhri* was analyzed by Woepcke (*Extrait du Fakhrī*, Paris, 1853), and Leonard of Pisa (1202). Nor must it be understood that there is any suggestion here of plagiarism, for undoubtedly at the time that Al-Karkhi wrote, Abu Kamil's methods were so well known that any writer could employ his results as common property, while the phraseology of Leonard of Pisa, in the statement of many problems drawn from Abu Kamil, is plainly intended to carry the idea of a citation. Al-Karkhi reproduced not only many problems given by Abu Kamil but also the geometrical solutions devised by this writer, while Leonard of Pisa drew copiously from Abu Kamil for the problems in the section of his *Liber Abbaci* entitled "Expliciunt introductiones algebre et almuchabale; Incipiunt questiones eiusdem."[4]

Abu Kamil's Works. Abu Kamil Shoja' ben Aslam ben Mohammed ben Shoja, the reckoner from Egypt, was an excellent and learned arithmetician. He wrote: The book of fortune, The book of the key to fortune, The book on algebra, The book of extracts, The book of omens (by the flight of birds), On

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[1] Karpinski, "Robert of Chester's Translation of the Algebra of Al-Khowarizmi," *Bibliotheca mathematica*, third series, XI, 125–131.

[2] Karpinski, "The Algebra of Abu Kamil Shoja'ben Aslam," *Bibliotheca mathematica*, third series, XII (1912), 40–55. To this article the reader is referred for more complete bibliographical references and for citations from the Latin text.

[3] H. Suter, in a personal communication to the author.

[4] *Scritti di Leonardo Pisano*, published by Prince Boncompagni, Rome, 1857, I, 410–459.

completion and diminution, The book on the rule of double false position, The book of surveying and geometry, The book of the adequate (possibly in arithmetic).[1] Such is the account of the life and works of Abu Kamil as given in the Kitab al-Fihrist (987 A. D.) of An-Nadim, who includes this writer in his list of the later (*i. e.*, nearly contemporary with An-Nadim) reckoners and arithmeticians. Aside from the works included in this list a treatise on the mensuration of the pentagon and the decagon exists in Latin and Hebrew translations, and a further arithmetical work is preserved in the original Arabic as well as in Latin and Hebrew. The work on the pentagon and decagon has been published in Italian by Sacerdote,[2] and in German by Suter.[3] The arithmetical work, in which he deals with first degree indeterminate equations, has been translated by Suter.[4] Other information about the life and activity of this great Arab we do not have. The titles mentioned indicate that, like so many of the later scientists in Europe, Abu Kamil was interested in magic, doubtless including astrology, and in omens.

The Sources of Information. The manuscript upon which I have based my study of Abu Kamil's algebra is the Paris manuscript 7377A of the Bibliothèque nationale.[5] The treatise covers folios 71^v–93^v, and contains between 35,000 and 50,000 words.[6]

In the opening sentence the author refers to his famous predecessor in this field, Mohammed ibn Musa al-Khowarizmi, and a little later again refers to the algebra of the same writer. Several references are made to Euclid but aside from these two names no others have been noted in this algebra.

Quotations from the Manuscript. The following translation of selected passages from the algebra is a very free one, as an attempt is made to preserve the mathematical significance. Additions to clarify the meaning, and modern notations, are put in parentheses.

The first thing which is necessary for students of this science is to understand the three species which are noted by Mohammed ibn Musa al-Khowarizmi in his book. These are *roots, squares* and *number*. A root is anything which can be multiplied by itself, composed of one, and numbers above one, and fractions. A square is that which results from the multiplication by itself of root, composed of units, and units and fractions. A number is a quantity by itself which does not have the name of root or square, but is proportional to the number of units which it contains. These three species are proportional to each other by turn in twos. Thus, squares equal to roots, squares equal to number, and roots equal to number ($ax^2 = bx$, $ax^2 = n$, $bx = n$).

An illustration of squares equal to roots is this:

A square is equal to five of its roots ($x^2 = 5x$). The explanation of this is that a square is five times the root of it. The root of the square is always the same as the number of the roots to which the square is equal. In this question this root is five and the square is twenty-five

[1] I translate from the German translation of the passages in the Fihrist dealing with mathematical scientists, by Suter, in *Abhandl. z. Geschichte d. math. Wissen.*, VI, 1–87.

[2] *Festschrift Steinschneiders*, Leipzig, 1896, 169–194.

[3] *Bibliotheca mathematica*, X, third series, 15–42.

[4] *Das Buch der Seltenheiten der Rechenkunst*, in *Bibl. math.*, 3d ser., XI, 100–120.

[5] I am indebted to the librarian for permission to have photographic reproductions made.

[6] To give an idea of the extent of this work, if the Latin text were printed in the MONTHLY it would take between 75 and 100 pages.

which is the same as five of its roots. That the root of the square is the same as the number of the root this I will explain.

We place for the square a square surface *abgd* whose sides are *ab*, *bg*, *gd*, and *da* (see Fig. 1). Of this square any side multiplied by one in number is the same as the area of a surface whose length is the side or root of the square. The side *ab* multiplied by one which is *be* gives the surface *ae* which is the root of the square *ag*. The surface *ag* is the same as five roots or five times the root of itself. Therefore, it is five times the surface *ae*. Divide then, the surface *ag* by equidistant lines into five equal parts, that is the surfaces *ae*, *ck*, *sr*, *nh*, and *mg*. The lines *be*, *ek*, *kr*, *rh* and *hg* are equal. Moreover the line *be* is one. Therefore the line *bg* will be five which is the root of the square and the square itself is twenty-five which is the surface *ag*. This is what we wished to explain.

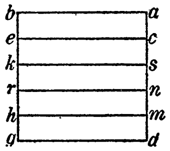

FIG. 1.

If it is proposed that one half a square should equal 10 roots ($\frac{1}{2}x^2 = 10x$) then the whole square is 20 roots. The root of the square is 20 and the square 400. Similarly if it is proposed that 5 squares equal 20 roots ($5x^2 = 20x$) then the root of the square is 4 and the square is 16. And whether more than a square or less than a square (is proposed) you reduce to 1 square. You operate in the same manner with that which is connected with the square, the roots.

A square which is equal to a number is illustrated:

A square equals 16 ($x^2 = 16$). Therefore, it itself is 16 and the root of it is 4. Similarly if given 5 squares equal 45 drachmas ($5x^2 = 45$), then 1 square is a fifth of 45 which is 9 and the root of it is 3. . . . So whether more or less than a square (is given) it is reduced to 1 square and similarly is treated that which is joined or equal to them of numbers.

Roots which are equal to numbers are illustrated:

A root equals 4 ($x = 4$). Therefore, the root itself is 4 and the square is 16. Similarly again if it is proposed that 5 roots should equal 30 ($5x = 30$) then the root is equal to 6 and the square 36. . . .

We construct now these three species, that is roots, squares and numbers, joined by twos and proportional to the third. This gives, a square and roots equal to number, a square and number equal to roots, and roots and number equal to squares.

(In modern notation $ax^2 + bx = n, \quad ax^2 + n = bx, \quad bx + n = ax^2$.)

First Type of Quadratic Equation. An illustration of squares and roots equal to number is the following:

A square and 10 roots equal to 39 drachmas ($x^2 + 10x = 39$). The explanation of this is that there is some square to which if you add the same as 10 of its roots the number of it results in 39 drachmas. In this question there are two ways (methods) of which one leads you to know the root of the square and the other to know the square of the root. We will set forth and explain each of these by a geometrical figure which will be understood by those who understand the book of Euclid, that is the Elements. The method which leads us to know the root of the square has been narrated by Mohammed ibn Musa al-Khowarizmi in his book. This is that you divide the roots in half, that is in two equal parts. In this question this gives 5. You multiply this by itself, giving 25, which you add to 39. This gives 64 of which number you take the root which is 8; from this you take the half of the roots, that is 5, and there remains 3, which is the root of the square. The square is 9.

The method indeed which leads you to know the square is that you multiply 10 roots by itself, giving 100. This you multiply by 39, which is equal to the square and roots, giving 3,900. Now divide 100 in halves, giving 50, which you multiply by itself, giving 2,500. This add to 3,900, giving 6,400, and of this number take the root, which is 80. This you subtract from 50, which is the half of 100, and from 39, which is equal to the square and the roots, together giving 89, and 9 remains, which is the square.[1]

[1] The text to this point is found in the Paris MS. on folio 71^{v}–72^{n}; given in my article in the *Bibliotheca Mathematica*, *loc. cit.*, p. 42–44.

In modern notation the steps of this solution are indicated as follows, in which both the general case and this special problem are given in parallel columns.

$$(1) \qquad x^2 + bx = n, \qquad\qquad x^2 + 10x = 39,$$

$$(2) \qquad b^2x^2 + b^3x = nb^2, \qquad\qquad 100x^2 + 1000x = 3{,}900,$$

$$(3) \qquad b^2x^2 + b^3x + \left(\frac{b^2}{2}\right)^2 = nb^2 + \frac{b^4}{4}, \qquad\qquad 100x^2 + 1000x + 2500 = 6{,}400,$$

$$(4) \qquad bx + \frac{b^2}{2} = +\sqrt{nb^2 + \frac{b^4}{4}}, \qquad\qquad 10x + 50 = 80.$$

Only the positive root is taken, as the negative root leads to the negative solution of the equation. Subtracting (4) from (1), member for member,

$$x^2 - \frac{b^2}{2} = n - \sqrt{nb^2 + \frac{b^4}{4}}, \qquad x^2 - 50 = 39 - 80,$$

$$x^2 = n + \frac{b^2}{2} - \sqrt{nb^2 + \frac{b^4}{4}}, \qquad x^2 = 50 + 39 - 80,$$

$$x^2 = 9.$$

Following this the problem is proposed:

$$2x^2 + 10x = 48.$$

This is a problem which is also given by Al-Khowarizmi and the same is true of many further problems given by Abu Kamil. The latter author gives the method of solution presented by Al-Khowarizmi and also the method, above illustrated, leading to the value of x^2 directly. Further than this he presents the geometrical solution for the first method as given by his more famous predecessor. Abu Kamil adds the geometrical solution corresponding to his own method, the second of those given above. This seems to be worth presenting here.

In order to show you the method (the door, literally) which leads to a knowledge of the square we place for the square the line *ab* (see Fig. 2). To this we add 10 roots of itself which are represented by the line *bg*, giving the line *ag*, which is 39. We wish to know then the value of the line *ab*. We construct then upon the line *bg* a square surface, which is the surface *degb*. This then will be $100x^2$ (in modern notation), that is, 100 times the line *ab*. . . . We construct then the surface *ah* the same as the square surface *be*, that is equal to the square, which is the same as the multiplication of the line *ab* in one of the units which it contains taken 100 times. Then we complete the surface *an* which is 3,900, since the line *ag* is 39 and *am* is 100, which lines contain this surface. Moreover the surface *ah* is the same as the surface *be*. Therefore, the surface *dn* is 3,900, which is produced by the multiplication of the line *ne* by *eg*, as the line *eg* is the same as the line *ed* and the line *gn* is 100 since it is the same as the line *am*. Therefore we divide the line *gn* in two equal parts at the point *l*, to which line so divided the line *ge* is added in length. Therefore, the multiplication of the whole line *ne* by *eg* together with the product of the line *gl* by itself is the same as the product of the line *le* by itself, as Euclid says in the second book[1] of his Elements. The product of *ne* by *eg* is 3,900 and the product of *gl*

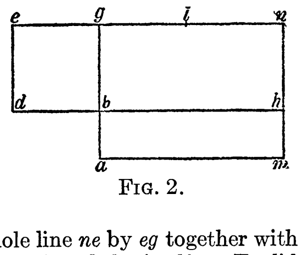

FIG. 2.

[1] Euclid's *Elements*, II, 6, "If a straight line is bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line, together with the square on the half, is equal to the square on the straight line made up of the half and the added straight line."

by itself is 2,500, which being combined make 6,400, which is the same as the product of the line *le* by itself. Therefore, the product of the line *le* by itself is 6,400, of which the root is 80, which is the line *le*. But the line *ge* is the same as the line *gb*, and so the line *lg* and the line *gb* make 80. When we take away the line *lg* and the line *dg*, which are 80, from the lines *ag*, 39, and *gl*, 50, which make 89, the line *ab* remains, which is 9. This is the square and that is what we desired to explain.[1]

Second Type of Quadratic Equations. The first problem which Abu Kamil presents of the second type of composite quadratic equations is again the same as that given by Al-Khowarizmi,

$$x^2 + 21 = 10x.$$

For this type of equation also Abu Kamil presents two solutions, the one leading directly to the value of the root of the equation and the other to the value of the square of the root. Further the author notes that there are two solutions for both the root and the square in this type of quadratic equation. The reason that this was noted by both of these Arabic writers is that in this type of equation the two roots are positive, whereas in the other two types the one root is positive and the other root is negative. Negative quantities as such are not accepted by these Arabic mathematicians and this, of course, accounts for the three types of quadratic equations. Express mention is also made of the fact that if the square of half the coefficient of the roots is less in magnitude than the constant then the problem is impossible, from the Arabic standpoint, or contradictory. Further also, the author mentions that when this square is equal to the given constant the root of the equation is the same as one half the given coefficient.

In the above problem,

$$x^2 + 21 = 10x,$$

the analytical solution leading to the value of the square of the unknown quantity proceeds in substance as follows:

Multiply 10 by 10, giving 100. Multiply this by 21, giving 2,100. Then take half of 100, giving 50, which you multiply by itself, giving 2,500. From this subtract 2,100, leaving 400 of which the root is 20, which you subtract from 50, the half of 100, leaving 30. From this you subtract 21, leaving 9, which is the square. And if you wish, add 20 to 50, giving 70, from which you subtract 21, leaving 49, which is the square and the root of it is 7.[2]

Al-Khowarizmi gave the geometrical figure corresponding only to the smaller root of the equation. Thus in the equation, $x^2 + n = bx$, in which

$$x = \frac{b}{2} \pm \sqrt{\frac{b^2}{4} - n},$$

one of these roots, when both are real, is necessarily less than $b/2$ and the other is greater. Abu Kamil gives the geometrical explanation corresponding to both, thus completing the work of Al-Khowarizmi. Thus he says that when you assume that the square is less (in numerical value) than the number which is given, then the solution appears by subtraction, and when you assume the same

[1] In the manuscript folio 72^v–73^r; in *Bib. Math., loc. cit.*, p. 46.

[2] Manuscript folio, 73^r; *Bib. Math., loc. cit.*, p. 47.

greater than the constant the solution appears by addition. Further he gives a geometrical solution for the type in which the square of the coefficient of x is equal to the given constant. The figure in this case consists simply of two equal squares with a side in common. We follow with a demonstration, original with Abu Kamil, leading to value of x^2 in the equation, $x^2 + 21 = 10x$.

. . . Place the line *ab* to represent x^2 (Fig. 3), and we add to it the drachmas which accompany it, 21, and let it be the line *bg*. Therefore, the line *ag* is 10 roots of the line *ab*. We construct then upon the line *ag*, a square surface, the square *aged* and this is 100 times the line *ab* . . . since the line *ag* is 10 roots of the line *ab*, and 10 roots multiplied by itself gives 100 squares (x^2). We construct then the surface *ah* equal to the square *aged* and let one side of it be the line *ab*. Therefore, the other side *bh* is 100. Then we complete the surface *an* and therefore the surface *bn* is 2,100 since the line *bg* is 21 and the line *gn* 100. We construct then the surface *my* equal to the surface *ae*, but the surface *ae* is equal to the surface *ah*. Take away, then, the surface *th* and there remains the area *cn* equal to the area *ca*. Adding then the surface *by*, the whole surface *ay* is equal to the surface *bn*, which is 2,100. Therefore the area *ay* is 2,100 which is the product of *gy* by *yn*, since *yn* is equal to *yt*, since the area *tn* is a square. We divide then the line *gn* in two equal parts at the point *l*. Therefore the line *ng* is divided into two equal parts by the point *l* and into two unequal parts by the point *y*. Therefore the product of *gy* by *yn* together with the square upon *ly* is the same as the product of *ln* by itself.[1] But *ln* by itself makes 2,500, since it is 50, and the product of *gy* by *yn* is 2,100. The square on the line *ly* is then left as 400 and the line *ly* 20. But *ln* is 50 and as it (*ly*) is to be subtracted the line *yn* is left as 30. Since the line *bg* is 21 the remaining line *ab* is 9 which is the square. This is what we wished to explain.[2]

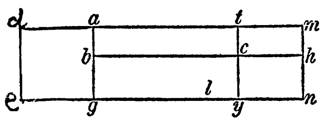

FIG. 3.

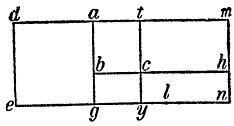

FIG. 4.

Having given the explanation "by subtraction," that is to say, when the square root of 2,500 − 2,100 is subtracted from 50, Abu Kamil proceeds to explain "by addition." *gn* is again divided into two equal parts but the point of division falls between *y* and *n* (Fig. 4). This corresponds to the assumption that the square is greater than the number which accompanies it. This has also been noted above.

The product of *gy* by *yn* together with the square of *yl* equals the product of *lg* by itself. *lg* multiplied by itself gives 2,500 and the line *yn* by *yg* gives 2,100. Therefore the square of *ly* is 400 and *ly* is 20. Since *ln* is 50, the whole line *yn* is 70. But *yn* is the same as *yt* and *yt* equals *ag*. Therefore *ag* is 70. But *bg* is 21 whence *ab* is 49 which is the square. This is what we desired to explain.[3]

In the third type of quadratic equations Abu Kamil again follows Al-Khowarizmi in taking the equation $3x + 4 = x^2$. Three explanations are presented for finding the root of the square. The first is equivalent to the demonstration presented by Al-Khowarizmi but the figure is not completed. Instead of this the proof is made to depend upon the second book of Euclid. The second gives

[1] This is by Euclid VI, 5, "If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole, together with square on the straight line between the points of section, is equal to the square on the half."

[2] Manuscript folio, 74^r; *Bib. Math.*, *loc. cit.*, pp. 48–49.

[3] Manuscript folio, 74^r; *Bib. Math.*, *loc. cit.*, p. 49.

the completed figure and explains as in the algebra of Al-Khowarizmi. In his third figure (Fig. 5) the square *abdg* is placed to represent x^2 which by the conditions of the problem equals $3x + 4$. Then the lines *ah* and *yd* are taken $1\frac{1}{2}$ units in length and the rectangles completed cutting off from the original square $3x$ less the little square, $1\frac{1}{2}$ on a side, which these two rectangles have in common. Hence the square which remains *hnyz* is $4 + (1\frac{1}{2})^2$ or $6\frac{1}{4}$. It follows that *gh* is $2\frac{1}{2}$ and *ay* (or x) is 4.

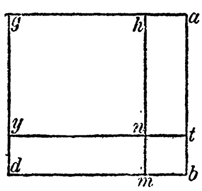

FIG. 5.

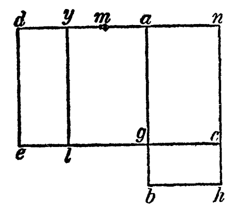

FIG. 6.

To arrive at the value of x^2 geometrically Abu Kamil places (Fig. 6) *ab* for the square and cuts off *gb* equal to 4, leaving *ag* equal to the three roots. Then he continues:

Upon *ag* a square is constructed *aged* whose area is $9x^2$ or 9 times the line *ab*. *ah* is constructed equal to the square *ae*. From this it follows that *an* is 9 and that *gc* which equals *an* is 9. Since *bg* is 4 the area of *gh* is 36. Take *ay* equal to *an* and draw *yl* parallel to *de*. The area *yg* equals *ac* and the area *ae* equals *ah*. Hence *ye* equals *cb* which is 36. But *ya* is 9 since *an* is 9. Divide *ay* into two equal parts by the point *m*. Now *yd* is added in length to *ay*. Hence *ad* by *dy* plus *ym* multiplied by itself equals *md* multiplied by itself, as EUCLID says in his second book. But *ad* by *dy* is *ye* or 36. *ym* multiplied by itself is $20\frac{1}{4}$. Adding you obtain $56\frac{1}{4}$. Therefore *md* multiplied by itself gives $56\frac{1}{4}$ and *md* is $7\frac{1}{2}$. But *am* is $4\frac{1}{2}$ and *gb* is 4. Hence *ab* is 16 and this is what we desired to demonstrate.[1]

Abu Kamil summarizes the results of the algebra up to this point, stating that all questions that can be solved by *algebra and almuchabala*, that is, restoration and opposition, must reduce to one or the other of the six types of equations mentioned.

Multiplication and Division. In the next section the multiplication and division of algebraic quantities is considered. Binomials, involving both addition and subtraction, are given particular consideration. The writer uses *res* and *radix*, indifferently, for the first power of the unknown. This is combined by addition or subtraction with a pure number. The second power of the unknown is designated by *census*, but this word is sometimes used simply for any unknown quantity as in problems below. Similar usage obtains in the work of Leonard of Pisa and in the *Liber augmenti et diminutionis* by one Abraham, published by Libri, *Histoire*, I, 304–371.

After a general statement about multiplication which summarizes the work presented, Abu Kamil continues with a geometrical explanation of the problems. Thus a figure containing four small squares is used to illustrate the fact that $2x$ multiplied by $2x$ gives $4x^2$. The next problem is $3x$ multiplied by 6 drachmas.

[1] Manuscript folio, 75^r; *Bib. Math.*, *loc. cit.*, pp. 49–50.

In this section the constant is referred to drachmas although in other places the word *numerus* is also used. Figures are shown for $10 + x$ by x, $10 - x$ by x, $10 + x$ by $10 - x$ and other similar examples.

Radical Expressions. The addition and subtraction of radicals, involving quadratic irrationalities only, is effected by means of the relations given in modern symbols by the equalities $\sqrt{a} \pm \sqrt{b} = \sqrt{a + b \pm 2\sqrt{ab}}$. Thus, to subtract the square root of 8 from the square root of 18 the rule is simply: "subtract 24 from 26, as you know, and 2 remains. The root of this is the root of 8 subtracted from the root of 18." This problem, by the way, is given by Al-Karkhi. Leonard of Pisa[1] employs the same method but uses 18 and 32, instead of 8 and 18, combining them by addition and subtraction in the same way as Abu Kamil.

Al-Karkhi[2] proceeds to the consideration of cube roots in a similar way but our author takes up for a final problem in roots the addition of $\sqrt{10}$ to the $\sqrt{2}$ which, he states, does not lead to a simple result since $10:2 = 5$ which is not a square anld aso $2:10 = \frac{1}{5}$ which is not a square. This section concludes with the statements that $\sqrt{10}+\sqrt{2}=\sqrt{12+2\sqrt{20}}$ and $\sqrt{10}-\sqrt{2}=\sqrt{12-2\sqrt{20}}$.

Problems. The section dealing with problems is introduced in much the same way that Al-Khowarizmi does the corresponding section, and many of the problems are taken from the older work. The first problem is to divide ten into two parts such that the square of the larger part should equal $1\frac{1}{2}$ times the product of the two parts. This leads to the equation

$$x^2 = 1\tfrac{1}{2}x(10 - x),$$

or

$$2\tfrac{1}{2}x^2 = 15x.$$

Thus the problem leads to the first of the six types of quadratic equations.

The above problem is the second presented by Leonard of Pisa[3] in the section with the heading: *Expliciunt introductiones algebre et almuchabale. Incipiunt questiones eiusdem.* Furthermore this problem does not appear in Al-Karkhi[4] in the list given by Woepcke. The conclusion is reasonable that Leonard had some source of information concerning the algebra of Abu Kamil other than Al-Karkhi's work, and even more particularly in view of the long series of problems found in Leonard's work which correspond to those given by Abu Kamil. The second and third questions here are found to be the same as the following two questions in Leonard of Pisa,[5] but with slightly different numbers. Thus instead of $6\frac{1}{4}x^2 = 100$, Leonard and Al-Khowarizmi give $2\frac{7}{9}x^2 = 100$ and in place of $x/(10 - x) = 4$ as given by Abu Kamil and Al-Khowarizmi, Leonard sets this same fraction equal to $2\frac{1}{3}$. Our space does not permit an examination of all similarities of this kind.

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[1] *Scritti di Leonardo Pisano*, published by Boncompagni, Vol. I: *Il liber abbaci*, Roma, 1857, p. 363–365.

[2] F. Woepcke, *Extrait du Fakhri*, Paris, 1853, p. 57–59.

[3] *Liber abbaci*, p. 410.

[4] Woepcke, loc. cit., p. 75–137.

[5] *Liber abbaci*, p. 410.

A problem given on folio 81^r is of particular interest because of its similarity to a problem of Al-Khowarizmi's[1] which has given rise to discussion. The problem is to divide 50 among some number of men and then to add three men and again distribute 50 (drachmas) among them (equally). In the second instance each man receives $3\frac{1}{2}\ \frac{1}{4}$ (meaning $3\ 1/2 + \frac{1}{4}$) drachmas less than in the first. Leonard states the corresponding problem as follows:

> Distribute 60 among (a certain number of) men and each will receive something. Add two men and among them all again distribute 60 and then each man will receive $2\frac{1}{2}$ denars less than at first.

The statement of Abu Kamil is:

> And if we tell you, distribute 50 among (a certain number of) men and each receives a certain amount (*res*). Add three men and distribute again 50 drachmas among them. Each then receives $3\frac{1}{2}\ \frac{1}{4}$ drachmas less than before.

Both follow with a geometrical explanation.

Problems Involving Fractions. The notation of fractions employed by this translator of Abu Kamil resembles the peculiar system employed later by Leonard. Thus on fol. 83^r the square of $\frac{1}{2}$ and of $\frac{1}{9}$ are given in combinations as in the following translation:

> . . . $\frac{1}{2}$ and $\frac{1}{9}$ which multiplied by itself, gives $\frac{1}{4}$ and $\frac{1\ 1}{9\ 9}$. To this we add 11 drachmas and $\frac{1}{9}$, giving 11 drachmas and $\frac{1}{4}$ and $\frac{1}{6}$ and $\frac{e\ 1}{9\ 2}$ and $\frac{e\ 1}{9\ 9}$. Of this we take the root or 3 and $\frac{1}{3}$ and $\frac{0\ 1}{9\ 2}$ . . . .

The fraction $\frac{1\ 1}{9\ 9}$ stands for $\frac{1}{9}$ plus $\frac{1}{81}$ whereas $\frac{e\ 1}{9\ 2}$ as well as $\frac{0\ 1}{9\ 2}$ represents simply $\frac{1}{18}$. So also Leonard[2] uses $\frac{5\ e}{8\ 8}$ for $\frac{5}{64}$. This system is explained by Leonard[3] who writes for $\frac{1}{14}$ the form $\frac{1\ 0}{2\ 7}$ and $\frac{1\ 5\ 7}{2\ 6\ 10}$ for $\frac{7}{10} + \frac{5}{60} + \frac{1}{120}$. The difference is only in the order of reading.

Al-Khowarizmi gives several problems of this nature:[4]

> To find a square of which if one-third be added to three dirhems, and the sum be subtracted from the square, the remainder, multiplied by itself restores the square.

The solution involves the use of x (root) to represent the square. So also Leonard[5] in a similar problem states: ". . . place for that square x (*res*)." Our version of Abu Kamil has several problems of the same kind:

> And if we tell you there is a quantity (*census*) from which when you subtract $\frac{1}{3}$ of itself and 2 drachmas, the product of the remainder by itself gives the quantity and 24 drachmas.

The solution is obtained by placing x (*res*) for the quantity (*census*).

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[1] Libri, *Histoire des sciences mathématiques en Italie*, Vol. 1, Paris, 1838: "Liber Maumeti . . . de algebra et almuchabala," p. 286. Rosen, *The Algebra of Mohammed ben Musa*, London, 1831, p. 63–64. Al-Khowarizmi does not present any geometrical explanation.

[2] *Liber abbaci*, p. 447.

[3] *Liber abbaci*, p. 24, 25.

[4] Rosen, *loc. cit.*, p. 56, 57.

[5] *Liber abbaci*, p. 422.

The longest discussion of any algebra problem in Leonard is that of the following:[1]

Divide 10 in two parts, and divide the larger by the smaller, and the smaller by the larger. Sum the results of the division and this equals the square root of 5.

Leonard presents several solutions. In the first he arrives at the equation $\sqrt{5x^4} + 2x^2 + 100 = 20x + \sqrt{500x^2}$. The coefficient of x^2 is made equal to unity by multiplying through by $\sqrt{5} - 2$. The same equation is found in Abu Kamil and the treatment is the same. The value of x is $5 - \sqrt{225 - \sqrt{50000}}$ which is found by both writers. Leonard goes on to find the value of $\sqrt{225 - \sqrt{50000}}$ as $\sqrt{125} - 10$. Al-Khowarizmi and Al-Karkhi do not give this problem although they do give one upon which this is based, namely: to divide 10 into two parts such that the sum of each divided by the other is $2\frac{1}{6}$. Al-Karkhi[2] discusses four solutions.

Leonard of Pisa occasionally solves incorrectly, if this expression be permissible. Thus the problem:[3] To find a number such that, if the square root of 3 be added to it and then the square root of 2 be added, the product of the two sums will be 20. Algebraically $(x + \sqrt{3})(x + \sqrt{2}) = 20$. The product of the two binomials is given as $x^2 + 6 + \sqrt{12x^2} + \sqrt{8x^2}$. This leads to the incorrect value $\sqrt{19 + \sqrt{24}} - \sqrt{3} - \sqrt{2}$ for x. Our manuscript of Abu Kamil gives the correct value, $\sqrt{21\frac{1}{4} + \sqrt{1\frac{1}{2}} - \sqrt{6}} - \sqrt{\frac{3}{4}} - \sqrt{\frac{1}{2}}$, for x. The problems[4] following this in the algebra of Leonard, with the exception of the two final ones, are all taken from Abu Kamil and in the same order in which they occur in the work of the Arabic writer.

The Italian writer frequently uses an expression which might be supposed to refer to Abu Kamil: "Operate according to algebra,"[5] . . . "Operate therefore in this according to algebra, etc." However our manuscript employs a similar expression. Thus in a problem arriving at $16x^4 = 256x^2$, this Latin version of Abu Kamil reads: Fac secundum algebra in eis et erunt 16 census census equales 256 censibus. census census (x^1) ergo equatur .16 censibus et census equatur 16 dragmis, ergo res equatur 4 dragmis . . ." (fol. 91^r). In the discussion of the same problem Leonard concludes one form of solution with the words: "Age ergo in eis secundum algebra, et inuenies, census census equari 16 censibus: quare census est 16, et radix eius est 4, ut dictum est." Both writers present several solutions, including a geometrical one, of the problem in question which is to divide 10 into two parts such that if from the larger you subtract two of its roots and to the smaller add two of its roots the quantities are then equal. The expression *secundum algebra* refers to the Book of Algebra and Almucabala by Mohammed ben Musa.

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[1] *Liber abbaci*, p. 434–438.
[2] Woepcke, *loc. cit.*, p. 91, 92.
[3] *Liber abbaci*, p. 445.
[4] *Liber abbaci*, 445–459.
[5] *Liber abbaci*, 438.

Treatise on the Pentagon and Decagon. The algebra terminates with the first eight lines on fol. 93v, and is followed by the work on the pentagon and decagon by the same author. Although this treatise on the pentagon and decagon by Abu Kamil is geometrical in its nature yet the treatment and the solutions are algebraical, including a fourth degree equation ($x^4 = 8000x^2 - \sqrt{51\,200\,000}$) as well as mixed quadratics with irrational coefficients. The twelfth problem is in the Latin: "Et si dicemus tibi trianguli equilateri et equianguli mensura est cum perpendiculari ipsius est 10 ex numero, quanta sit perpendicularis?"[1] This suggests the similar problems in Greek of unknown date and author presented by Heiberg and Zeuthen,[2] as well as the similar problems given by Diophantos. In these Greek problems also lines and areas are summed quite contrary to ancient Greek usage. A further point of interest is that in the equation $x^2 + 75 = 75x$, to which the solution of the thirteenth problem leads, Abu Kamil gives only one solution whereas in the algebra he recognizes that this equation has two positive roots. The opening sentence of this treatise on the pentagon and decagon makes reference to the algebra as immediately preceding it, which indeed is the fact in the Hebrew and Latin manuscripts that are preserved.

Conclusion. The algebra terminates with a general statement to the effect that by the methods taught in this book many more problems can be easily solved. In true Arabic fashion, the closing words are: "Whence praise and glory be to the only Creator."

Let us summarize the results of our study. The most important conclusion of this investigation of Abu Kamil's algebra is that Al-Karkhi and Leonard of Pisa drew extensively from this Arabic writer. Through them this man, though himself comparatively unknown to modern writers, exerted a powerful influence on the early development of algebra. Abu Kamil deserves somewhat the same recognition from modern mathematicians and historians of science as that which Leonard of Pisa and Al-Khowarizmi have received. We may hope that the future will be more just than the past in according to Abu Kamil a prominent place among the mathematicians of the middle ages.

A CURIOUS CONVERGENT SERIES.

By A. J. KEMPNER, University of Illinois.

It is well known that the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges. The object of this Note is to prove that if the denominators do not

[1] "If we say to you that an equilateral and equiangular triangle, together with its altitude, is measured by 10, what is the altitude?"

[2] *Einige griechische Aufgaben der unbestimmten Analytik*, in *Biblioth. mathem.*, VIII, third series, 118–134.

include all natural numbers 1, 2, 3, $\cdots$, but only those numbers which do not contain any figure 9, the series converges. The method of proof holds unchanged if, instead of 9, any other figure 1, 2, $\cdots$, 8 is excluded, but not for the figure 0.

Proof: The series with which we deal is

$$\text{I.}\quad \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{8}+\frac{1}{10}+\cdots+\frac{1}{18}+\frac{1}{20}+\cdots+\frac{1}{28}+\frac{1}{30}+\cdots$$

$$+\frac{1}{38}+\frac{1}{40}+\cdots+\frac{1}{48}+\frac{1}{50}+\cdots+\frac{1}{58}+\frac{1}{60}+\cdots+\frac{1}{68}+\frac{1}{70}+\cdots$$

$$+\frac{1}{78}+\frac{1}{80}+\cdots+\frac{1}{88}+\frac{1}{100}+\cdots.$$

We form a new series

$$\text{II.}\qquad s = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

by the following rule: a_n is the sum of all terms in I of denominator d, where $10^{n-1} \leqq d < 10^n$. We have then

$$a_1 = \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{8},$$

$$a_2 = \frac{1}{10}+\cdots+\frac{1}{18}+\frac{1}{20}+\cdots, \text{ the last term being } \frac{1}{88}, \text{ etc.}$$

Each term of I which forms part of a_1, is $\leqq 1$,

each term of I which forms part of a_2, is $\leqq 1/10$, and

each term of I which forms part of a_n, is $\leqq 1/10^{n-1}$.

We now count the number of terms of I which are contained in a_1, in a_2, $\cdots$, in a_n. Evidently a_1 consists of 8 terms, and $a_1 < 8 \cdot 1 < 9$. In a_2 there are, as is easily seen, less than 9^2 terms of I, and $a_2 < (9^2/10)$. Altogether there are in I less than $9^2 + 9$ terms with denominators under 100. Assume now that we know the number of terms in I which are contained in a_n to be less than 9^n, for $n = 1, 2, 3, \cdots, n$. Then, because each term of I which is contained in a_n is not greater than $1/10^{n-1}$, we have $a_n < (9^n/10^{n-1})$, and the total number of terms in I with denominators under 10^n is less than $9^n + 9^{n-1} + 9^{n-2} + \cdots + 9^2 + 9$. For $n = 1$ and $n = 2$ we have verified all this, and we will now show that if it is true for n, then $a_{n+1} < (9^{n+1}/10^n)$. a_{n+1} contains all terms in I of denominator d, $10^n \leqq d < 10^{n+1}$. This interval for d can be broken up into the nine intervals $\alpha \cdot 10^n \leqq d < (\alpha + 1)10^n$, $\alpha = 1, 2, \cdots, 9$. The last interval does not contribute any term to I, the eight remaining intervals contribute each the same number of terms to I, and this is the same as the number of terms contributed by the whole interval $0 < d < 10^n$, that is, by assumption, less than

$9^n + 9^{n-1} + 9^{n-2} + \cdots + 9^2 + 9$. Altogether, therefore, a_{n+1} contains less than $8(9^n + 9^{n-1} + 9^{n-2} + \cdots + 9^2 + 9) < 9^{n+1}$ terms of I, and, as each of these terms is not greater than $1/10^n$, we have $a_{n+1} < (9^{n+1}/10^n)$.

Our series is therefore

$$s = a_1 + a_2 + a_3 + \cdots < 9 + \frac{9^2}{10} + \frac{9^3}{10^2} + \cdots + \frac{9^{n+1}}{10^n} + \cdots = 90,$$

and I is certainly convergent.

The preceding proof also contains all material necessary to show the following: Let M be any positive integer, and N_1 the number of positive integers $< M$ containing no figure 9, N_2 the number of positive integers $< M$ containing at least one figure 9, then

$$\lim_{M \doteq \infty} N_1/N_2 = 0.$$

See problem 207 proposed on page 55 of this issue.

THE PERFECT MAGIC SQUARES FOR 1914.

By V. M. SPUNAR, Chicago, Ill.

Using Zerr's formula*

$$\frac{1914 - (n^3 + n)/2}{n} = \text{Integer},$$

the only values of n for 1914 are 3, 11, 29. (For 1913 there are no values for n whatever.) The least and greatest integers permissible for forming the squares are represented by another formula of Zerr's, namely,

$$\frac{3828 \pm (n^3 - n)}{2n}.$$

For $n = 3$, the least integer $= 634$, the greatest integer $= 642$. For $n = 11$, the least integer $= 114$, the greatest integer $= 234$. For $n = 29$, the least integer $= -354$, the greatest integer $= 486$.

Hence the magic squares for 1914 with only positive numbers are as follows:

$n = 3$

641	634	639
636	638	640
637	642	635

* American Mathematical Monthly, Vol. XVI, no. 1, page 2.

$n = 11$

181	194	207	220	233	114	127	140	153	166	179
193	206	219	232	124	126	139	152	165	178	180
205	218	231	123	125	138	151	164	177	190	192
217	230	122	135	137	150	163	176	189	191	204
229	121	134	136	149	162	175	188	201	203	216
120	133	146	148	161	174	187	200	202	215	228
132	145	147	160	173	186	199	212	214	227	119
144	157	159	172	185	198	211	213	226	118	131
156	158	171	184	197	210	223	225	117	130	143
168	170	183	196	209	222	224	116	129	142	155
169	182	195	208	221	234	115	128	141	154	167

NOTE ON A MEMORY DEVICE FOR HYPERBOLIC FUNCTIONS.

By F. S. ELDER, Central High School, Kansas City, Mo.

Purely as a memory aid for expressing any hyperbolic function $\phi h(u)$ in terms of any other hyperbolic function $\psi h(u)$, I offer the following method which I never have seen in print.

Draw the customary right triangle ABC right angled at C, as for the circular functions. Write $\psi h(u)$ and unity upon the same sides of the triangle as you would to show the corresponding circular function. Then take the triangle by the point C and deform it until B becomes the right angle. In this deformation, the two sides on which the numbers are written are supposed to retain their original length, while the third side is deformed (lengthened or shortened) and likewise in returning to the original form. Now compute the third side in terms of the other two and let the triangle recover its original shape. All the other hyperbolic functions, $\phi h(u)$, can now be written down directly in terms of $\psi h(u)$ by means of the same ratios of the sides of the original triangle that give the corresponding circular functions.

Examples: (1) If $\psi h(u)$ be $\tan h(u)$, write $AC = 1$, $BC = \tan h(u)$. Deform. Then $AB = \sqrt{1 - \tan^2 h(u)}$; now restore the triangle.

Then

$$\sin h(u) = BC/AB = \frac{\tan h(u)}{\sqrt{1 - \tan^2 h(u)}}, \quad \sec h(u) = AB/AC = \sqrt{1 - \tan^2 h(u)}, \text{ etc.}$$

(2) If $\psi h(u)$ be $\csc h(u)$, write $AB = \csc h(u)$; $BC = 1$. Deform. Then $AC = \sqrt{1 + \csc^2 h(u)}$. Restore.

Then

$$\cos h(u) = AC/AB = \frac{\sqrt{1 + \csc^2 h(u)}}{\csc h(u)}, \quad \tan h(u) = BC/AC = \frac{1}{\sqrt{1 + \csc^2 h(u)}}, \text{ etc.}$$

BOOK REVIEWS.

UNDER THE DIRECTION OF W. H. BUSSEY.

Mathematik für Biologen und Chemiker. By Professor L. MICHAELIS, Private Docent at the University of Berlin. Springer, Berlin, 1912. vii+253 pages. 7.80 Marks.

This is an interesting example of a type of text similar to the English book by Partington,[1] designed as a minimum course in calculus for students whose engrossing interest is in other than mathematical lines. It is written in clear and "easy" German, and is brought thus to the attention of the readers of the Monthly because its perusal along with other similar texts cannot fail to be of value to those who, either inside or outside the ranks of teachers of mathematics, are concerned in bringing the benefits of the study of calculus to students of general science.

In the first fifty pages the author presents a compendium of the essentials of elementary geometry, arithmetic, algebra and trigonometry; he continues with chapters treating the theory of elementary functions (coördinate geometry), differential calculus (52 pp.), integral calculus (60 pp.), Maclaurin's and Taylor's series (19 pp.), and differential equations (28 pp.). Professor Michaelis frankly adopts the infinitesimal method of exposition *ab initio,* defining dx as an infinitely small increment given to x, adding the statement that dy is similarly the differential of y and that dy/dx is the differential quotient. As a consequence of this style of treatment, the author in his derivation of the differential quotient of the logarithm is spared the logical necessity of proving (or even of mentioning) that "the limit of the logarithm equals the logarithm of the limit."

Numerous applications of calculus to mechanics and chemistry increase the value of the text, although there is little of direct application to biology, as the title would seem to suggest. W. DEW. CAIRNS.

Tables and Formulas for Solving Numerical Problems in Analytic Geometry, Calculus and Applied Mathematics. Arranged by WILLIAM RAYMOND LONGLEY, assistant professor of mathematics in the Sheffield Scientific School, Yale University. Ginn and Company, Boston, 1913. iv+31 pages. $0.50.

The tables in this book comprise the usual four-place trigonometric and logarithmic tables, the equivalents in radians of angles measured in degrees, the natural functions of angles measured in radians, squares, cubes, square roots, cube roots, reciprocals, Naperian logarithms, and exponential and hyperbolic functions.

The formulas from algebra and elementary geometry number twenty, those from trigonometry twenty-two, while there are fifteen from plane and solid analytic geometry and eight from the calculus in addition to twenty-seven formulas for differentiation and one hundred and twenty-seven for integration.

The size of the book ($7\frac{1}{2}$ in. x 5 in.) and its well chosen material should make it popular with those dealing with computational work, especially in technical

[1] *Higher Mathematics for Chemical Students.* By J. R. PARTINGTON. New York, D. Van Nostrand Co., 1912.

schools for which the book was primarily prepared. The typographical clearness is especially to be commended. A. L. UNDERHILL.

Society for the Promotion of Engineering Education, Proceedings of the Twentieth Annual Meeting, June 26–29, 1912, Vol. XX, Part II. Published by the Secretary, Professor H. H. Norris, Ithaca, N. Y. xxii + 508 pages. $1.25 to non-members.

This volume is a splendid example of what may be done to share with all the members of a society and with others outside the society the full benefits of the annual meetings. According to the well-arranged statistics which form the introduction, the membership of the Society numbered 1,166, while only 215 members and guests attended the Boston meeting; yet the report of the treasurer shows that with an annual fee of four dollars the budget makes possible the distribution to members of the full proceedings of the annual meetings as well as of a monthly *Bulletin.* Why may not this be done in more of our American societies than is the case at present, whether these are directly educational or not? It is at times exceedingly difficult to obtain certain papers on a given program, inasmuch as the publishing of these is scattered over a wide range of journals. This *Bulletin* by the way affords an active forum for a live and even spicy discussion of matters bearing particularly on the teaching of engineering subjects, and enables a group of members to carry on during the year an exchange of opinions and a comparison of methods which cannot at all take place with such fullness and deliberation in connection with the all too crowded programs. The practical question fairly forces itself upon our attention as to the possibility of such interchanges on the part of those interested in the teaching of mathematics. A suggestion lies close at hand. Can this be done through the columns of the AMERICAN MATHEMATICAL MONTHLY?

Another valuable feature is the animated discussions which follow the presentation of a set of papers clustering about a common topic, as witness the shorter comments and somewhat more extended criticisms in the *Bulletin* called forth by a paper on "The Teaching of Elementary Physics" by Professors Franklin and MacNutt, or the discussion following a valuable report by Professor W. T. Magruder which collates the results of an investigation of the mechanical engineering laboratories in more than twenty-five institutions in this country with regard to equipment, personnel and methods. The remarks made are always "parliamentary" and in good spirit, but are given now and again "without gloves"; and woe be to the man who has a hobby, or who expresses his opinions either immoderately or vaguely!

Aside from the articles just referred to, there may be singled out for mention papers on "The Faculty Seminar" by Professor H. H. Norris, on "The Hydraulic Equipment at the Ohio State University" by Professor Horace Judd and on "The Engineering Laboratories of the Royal Technical University at Charlottenburg, Germany" by Mr. R. R. Heuter.

The volume may be obtained directly from the Secretary of the Society.

W. DEW. CAIRNS.

PROBLEMS AND SOLUTIONS.

B. F. Finkel, Chairman of the Committee.

PROBLEMS FOR SOLUTION.

Special Notice.—Please reread the requests as to form of solutions on pp. 258–259 of the October 1913 issue. Unless these directions are observed by contributors, solutions must either be entirely rewritten by the committee or else rejected. Put all drawings on separate sheets.

Managing Editor.

ALGEBRA.

When this issue was made up no solutions had been received for numbers 396 to 400 inclusive. Please give attention to these.

401. Proposed by R. D. CARMICHAEL, Indiana University.

Prove the validity of Borda's series:

$$\log(x+2) = 2\log(x+1) - 2\log(x-1) + \log(x-2)$$
$$+2\left[\frac{2}{x^3-3x} + \frac{1}{3}\left(\frac{2}{x^3-3x}\right)^3 + \frac{1}{5}\left(\frac{2}{x^3-3x}\right)^5 + \cdots\right].$$

402. Proposed by R. D. CARMICHAEL, Indiana University.

Obtain other series similar to that of Borda, given in the preceding problem.

403. Proposed by C. N. SCHMALL, New York City.

A torpedo-boat 40 miles from shore strikes a rock, making a rent in her hull which admits water at the rate of 15 tons in 48 minutes. The ship's pumps can expel 12 tons in an hour. If 60 tons of water is sufficient to sink the boat, find the average rate of steaming so that it may reach the shore just as it is about to sink.

404. Proposed by V. M. SPUNAR, Chicago, Illinois.

Show that $(a+b)(a+b-1)\cdots(a+b-n+1) = a(a-1)(a-2)\cdots(a-n+1) + \binom{n}{1}a(a-1)(a-2)\cdots(a-n+1)b + \binom{n}{2}a(a-1)(a-2)\cdots(a-n+1)b(b-1) + \cdots + b(b-1)(b-2)\cdots(b-n+1)$.

GEOMETRY.

When this issue was made up no solutions had been received for numbers 417, 421 and 425 to 430 inclusive. Please give attention to these.

431. Proposed by F. M. MORGAN, Dartmouth College.

Trisect the angles of the triangle ABC and let the trisectors nearest each side meet in the respective points M, N, P. Prove by trigonometry that the triangle MNP is equilateral.

432. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Having given the edges of a tetrahedron, a, b, c, d, e, f, find an expression for the radius of the sphere which is tangent to the six edges.

433. Proposed by W. H. BUSSEY, University of Minnesota.

A transformation of the plane keeping the radius of curvature of all curves invariant is either (1) a real or imaginary motion or reflexion, or (2) not a point transformation.

CALCULUS.

When this issue was made up no solutions had been received for numbers 335, 337, 338, 340, 342, 348, 350, and 352.

353. Proposed by RICHARD P. LOCHNER, Philadelphia, Pa.

The center of a sphere, radius $R = 5$ inches, is $a = 10$ inches above the surface of a sphere, radius $R = 12\frac{1}{2}$ inches. There is a point of light at $b = 1$ inch horizontally from a point $c = 10$ inches vertically above the surface of the first sphere. What is the area of the shadow which the upper sphere casts on the lower one?

354. Proposed by C. N. SCHMALL, New York City.

Prove $\Gamma(1 + x)\Gamma(1 - x) = \dfrac{\pi x}{\sin \pi x}$.

355. Proposed by C. N. SCHMALL, New York City.

Given the curve of the nth degree,

$$y^n - (a + bx)y^{n-1} + (c + dx + ex^2)y^{n-2} + \cdots = 0,$$

show that if each ordinate is divided by the corresponding subtangent, the sum of all the resulting ratios will be a constant.

MECHANICS.

When this issue was made up no solutions had been received for numbers 271, 272, 275, 277, 278, 279, and 282.

286. Proposed by C. N. SCHMALL, New York City.

A slightly elastic string is just long enough to reach between two hooks on the same horizontal line. A ring of weight w is placed at its middle point. Show that the ring will sink through a distance $h = a\sqrt[3]{3ew/2}$, where e is the elasticity of the string and $2a$ the distance between the hooks.

287. Proposed by WALTER H. DRANE, Lebanon, Tenn.

While sitting in an empaled enclosure, I noticed that the spokes of the wheels of passing automobiles, when viewed through the pickets of the fence, appeared to revolve more slowly than they really did, and in some instances even appeared to be revolving in a direction opposite to that in which they were really turning. Explain this optical illusion.

NUMBER THEORY.

When this issue was made up no solutions had been received for numbers 187, 189, 191, 192, 194, 201, and 202 inclusive.

205. Proposed by E. T. BELL, New York City.

Show that in the usual arithmetical sense the form that follows admits of composition; give the requisite transformations, and indicate how several (if not all) solutions may be found. The variables are the x_i.

$$x_0^2 + nrx_1^2 + mrx_2^2 + mnx_3^2 + mnrx_4^2 + mn^2r^2x_5^2 + nr^2m^2x_6^2 + rm^2n^2x_7^2.$$

206. Proposed by R. D. CARMICHAEL, Indiana University.

Prove that the sum of the sixth powers of two integers cannot be the square of an integer.

207. Proposed by A. J. KEMPNER, University of Illinois.

There are 80 positive integers < 100 containing no figure 9 against 19 containing at least one figure 9. (For integers < 1000 the numbers are 728 and 271 respectively.) One might be led to believe that for every positive integer M the number N_1 of positive integers $< M$ containing no figure 9 is always greater than the number N_2 of positive integers $< M$ containing at least one figure 9.

To prove: $\lim\limits_{M \doteq \infty} N_1/N_2 = 0$. See pages 48–50 of January, 1914, issue.

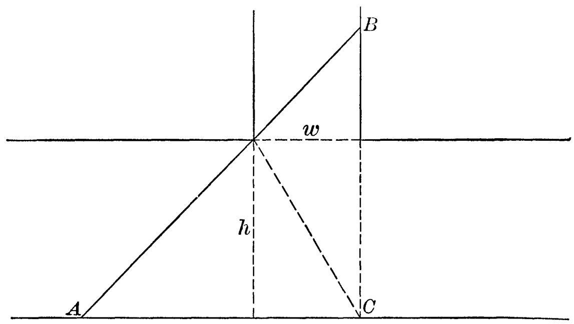
SOLUTIONS OF PROBLEMS.

ALGEBRA.

392. Proposed by H. PRIME, Boston, Mass.

The floor and ceiling of a room h feet high are parallel and horizontal. In the middle of the ceiling is a vertical circular tube w feet in diameter extending upward from the ceiling. From the room the longest possible inflexible rod is put up the tube. When the rod is in contact with the opposite sides of the tube and the floor, what is the expression for the tangent of the acute angle made by the rod and the floor in terms of h and w, considering the rod as an air line? To be solved without the calculus. (From The Maine Farmers' Almanac, 1913.)

I. Solution by H. E. Trefethen, Colby College.



If AB is the rod, $AC = x$ its projection on the floor, and $BC = y$, then

$$x^2 + y^2 = AB^2 = \text{minimum}, \qquad w/x + h/y = 1. \tag{1, 2}$$

Put $w/x = mu$, $x^2 = mu^{-2}$, $h/y = nv$, $y^2 = nv^{-2}$. Then $m = w^{\frac{2}{3}}$, $n = h^{\frac{2}{3}}$. Hence (1, 2) become $w^{\frac{2}{3}}u^{-2} + h^{\frac{2}{3}}v^{-2} = \text{minimum}$, $w^{\frac{2}{3}}u + h^{\frac{2}{3}}v = 1$.

Now

$$(w^{\frac{2}{3}}u^{-2} + h^{\frac{2}{3}}v^{-2})/(w^{\frac{2}{3}} + h^{\frac{2}{3}}) \geqq [(w^{\frac{2}{3}}u + h^{\frac{2}{3}}v)/(w^{\frac{2}{3}} + h^{\frac{2}{3}})]^{-2}. \tag{3}$$

For if

$$w^{\frac{2}{3}}/(w^{\frac{2}{3}} + h^{\frac{2}{3}}) = p, \qquad h^{\frac{2}{3}}/(w^{\frac{2}{3}} + h^{\frac{2}{3}}) = q, \quad \text{then} \quad p + q = 1, \tag{4}$$

and if

$$u/(pu + qv) = r, \qquad v/(pu + qv) = s, \quad \text{then} \quad pr + qs = 1, \tag{5}$$

and if we divide both sides of (3) by the second member, then (3) reduces to $pr^{-2} + qs^{-2} \geqq 1$, which is true since

$$1 - r^2 \geqq 2r^2(1 - r) \quad \text{and} \quad 1 - s^2 \geqq 2s^2(1 - s).$$

And hence

$$r^{-2} - 1 \geqq 2(1 - r) \quad \text{and} \quad pr^{-2} - p \geqq 2p - 2pr, \tag{6}$$

$$s^{-2} - 1 \geqq 2(1 - s) \quad \text{and} \quad qs^{-2} - q \geqq 2q - 2qs. \tag{7}$$

Adding (6, 7) and reducing by (4, 5) we have $pr^{-2} + qs^{-2} \geqq 1$.

Thus (3) is established. Therefore, since the denominators and the second

numerator are constants, $w^{\frac{2}{3}}u^{-2} + h^{\frac{2}{3}}v^{-2} =$ minimum when (3) is an equality, that is, when $u = v$. Hence $w\,/x = h^{\frac{1}{3}}/y$ and

$$y/x = h^{\frac{1}{3}}/w^{\frac{1}{3}} = \tan BAC \text{ as required.}$$

II. Solution by A. H. Holmes, Brunswick, Me.

A maximum length of pole would move with equal freedom on floor and side of tube, the condition being that the ratio between the part of the pole below the tube to the tangent of the angle it makes with the floor is equal to the ratio between the part of the pole within the tube and the tangent of the angle made by the extremity of the pole with the side of the tube, which is the cotangent of the angle made by the pole with the floor. Let h be the perpendicular height of the ceiling from the floor, w the diameter of the tube, and θ the acute angle made by the pole with the floor. Then we have

$$\frac{h}{\sin\theta} : \tan\theta = \frac{w}{\cos\theta} : \cot\theta.$$

Hence $\tan\theta = \sqrt[3]{h/w}$.

393. Proposed by H. E. TREFETHEN, Waterville, Maine.

Expand $\log[(\sin x)/x]$ in terms of x.

I. Solution by F. M. Morgan, Dartmouth College.

$$\sin x = x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \frac{x^7}{7\,!} + \cdots.$$

Hence

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3\,!} + \frac{x^4}{5\,!} - \frac{x^6}{7\,!} + \cdots.$$

Now let

$$\frac{x^2}{3\,!} - \frac{x^4}{5\,!} + \frac{x^6}{7\,!} - \cdots = \gamma.$$

Then

$$\frac{\sin x}{x} = 1 - \gamma, \quad \text{and} \quad \log(1-\gamma) = -\gamma - \frac{\gamma^2}{2} - \frac{\gamma^3}{3} - \cdots.$$

Hence

$$\log[(\sin x)/x] = -\left(\frac{x^2}{3\,!} - \frac{x^4}{5\,!} + \cdots\right) - \left(\frac{x^2}{3\,!} - \frac{x^4}{5\,!} + \cdots\right)^2 - \cdots$$

$$= -\frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \cdots.$$

II. Solution by A. H. Holmes, Brunswick, Me.

$$\frac{d}{dx}\log[(\sin x)/x] = \cot x - \frac{1}{x}.$$

But

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots.$$

Hence,

$$\frac{d}{dx}\log\,[(\sin x)/x] = -\frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots$$

from which by integration, we get

$$\log\,[(\sin x)/x] = -\frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \cdots$$

A solution similar to I was received from LEVY M. COFFIN.

GEOMETRY.

423. Proposed by C. N. SCHMALL, New York City.

The sum of the squares of the distances of a point from n fixed points is constant. Show that the locus of the point is a circle.

SOLUTION BY W. C. EELLS, Tacoma, Wash.

Let (x_1, y_1), $(x_2, y_2) \cdots (x_n, y_n)$ be the coördinates of the n fixed points, and (x, y) of the moving point. Since $d_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}$, the given condition may be expressed,

$$(x - x_1)^2 + (y - y_1)^2 + (x - x_2)^2 + (y - y_2)^2 + \cdots + (x - x_n)^2 + (y - y_n)^2 = k,$$

which reduces to

$$nx^2 + ny^2 - 2(x_1 + x_2 + \cdots + x_n)x - 2(y_1 + y_2 + \cdots + y_n)y$$
$$+ (x_1^2 + x_2^2 + \cdots + x_n^2) + (y_1^2 + y_2^2 + \cdots + y_n^2) - k = 0.$$

This may be expressed more briefly,

$$x^2 + y^2 - \left(\frac{2}{n}\sum_{i=1}^{n} x_i\right)x - \left(\frac{2}{n}\sum_{i=1}^{n} y_i\right)y + \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 - k\right) = 0.$$

This is a circle, with center at $\left(\frac{1}{n}\sum_{i=1}^{n} x_i,\ \frac{1}{n}\sum_{i=1}^{n} y_i\right)$, the square of whose radius is

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 + \left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 - k\right).$$

Similar solutions were received from H. C. FEEMSTER, S. W. REAVES, F. M. MORGAN, ELMER SCHUYLER, and C. N. SCHMALL.

A solution of 418 was received from D. F. KELLEY; one of 420 from C. N. SCHMALL, and one of 422 from H. E. TREFETHEN, after the list of solutions was made up and sent to the printer.

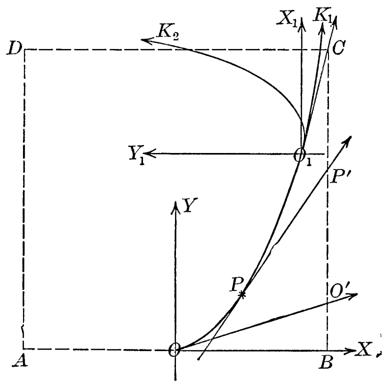
CALCULUS.

327. Proposed by RICHARD P. LOCHNER, Philadelphia, Penn.

A hound is at the middle point of the side of a square field, and a fox is at an adjacent corner. How far will the hound run to catch the fox if the fox runs on the perimeter of the field and the hound runs directly towards the fox at all times, the hound running n times as fast as the fox? Where will the race end?

SOLUTION BY H. EGERER, Wilkinsburg, Pa.

The hound is running on a curve, which has the property that the length of the curve between two points is proportional to the distance cut off by the tangents at these two points on a straight line.



To find this curve $y = f(x)$, which we call K, we take two perpendicular axes through a point O of the curve, so that the y-axis is parallel to the given straight line BC. Furthermore, O', the starting point of the fox, is given by $OB = a$, $O'B = b$. Then we have

$$\text{arc } OP = n \cdot O'P'.$$

Now

$$\text{arc } OP = \int_0^x \sqrt{1 + y'^2}\, dx, \qquad O'P' = y + y'(a - x) - b,$$

so that we have

$$y - b + y'(a - x) = \frac{1}{n}\int_0^x \sqrt{1 + y'^2}\, dx.$$

By differentiation, we get

$$y''(a - x) = \frac{1}{n}\sqrt{1 + y'^2}.$$

If we assume $y' = p$, hence $y'' = \dfrac{dp}{dx}$, we have $\dfrac{dx}{n(a - x)} = \dfrac{dp}{\sqrt{1 + p^2}}$, and by integration

$$c(a - x)^{1/n} = p + \sqrt{1 + p^2},$$

or

$$p = \frac{dy}{dx} = \frac{c}{2}(a - x)^{1/n} - \frac{1}{2c}(a - x)^{-1/n},$$

and again by integration

$$y = y_a + \frac{n}{2c(n - 1)}(a - x)^{(n-1)/n} - \frac{cn}{2(n + 1)}(a - x)^{(n+1)/n}.$$

Since when $x = y = 0$, $p = \dfrac{b}{a}$, therefore $c = \dfrac{b + \sqrt{a^2 + b^2}}{a^{(n+1)/n}}$.

Hence, also

$$y_a = \frac{cn}{2(n+1)} a^{(n+1)/n} - \frac{n}{2c(n-1)} a^{(n-1)/n}.$$

Therefore, finally, we have

$$y = \frac{cn}{2(n+1)} (a^{(n+1)/n} - (a-x)^{(n+1)/n}) - \frac{n}{2c(n-1)} (a^{(n-1)/n} - (a-x)^{(n-1)/n}).$$

We see that the curve K_1 cuts BC at the point (a, y_a) and, since for $n = 1$ $y_a = \infty$, it follows that n must be greater than 1 if the hound can catch the fox.

If we assume the side of the square AB equal to 2, then we have $a = 1$, $b = 0$ and therefore $c = \pm 1$, where we have to take $c = -1$, because y' is always positive and we get

$$y = \frac{n}{2(n-1)} (1 - (1-x)^{(n-1)/n}) - \frac{n}{2(n+1)} (1 - (1-x)^{(n+1)/n}),$$

and

$$y_a = \frac{n}{2(n-1)} - \frac{n}{2(n+1)}.$$

As long as $y_a < 2$, the hound catches the fox on the side BC of the square, and we find the smallest n from $2 = \dfrac{n}{2(n-1)} - \dfrac{n}{2(n+1)}$, that is, $n = 1.3680$. If $n < 1.3680$, then we find the point $O_1 \equiv (x_1, y_1)$ whose tangent passes through C and lay a new system of axes X_1Y_1 for the curve K_2. Here is $a = 2 - y_1$, $b = 1 - x_1$, so that we find the equation of the curve K_2 and can find y_a in which alone we are interested. If the hound does not catch the fox on the side CD we have to repeat the process.

We find O_1 by use of the equation $2 = y + y'(a - x)$.

By substituting the above values for y and y', we find

$$\frac{2n}{n^2 - 1} - 4 = \frac{(1-x)^{(n+1)/n}}{n+1} + \frac{(1-x)^{(n-1)/n}}{n-1}.$$

If we assume $(1 - x)^{1/n} = U$, we have the form $a = bU^m + cU^{m'}$ from which we find U, and hence x and O_1.

Also solved by BARNEM LIBBY.

334. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Solve

$$\frac{\partial^2 T}{\partial u \partial v} + \frac{2v}{u^2 + v^2 + 1} \frac{\partial T}{\partial u} + \frac{2u}{u^2 + v^2 + 1} \frac{\partial T}{\partial v} = 0.$$

SOLUTION BY R. D. CARMICHAEL, Indiana University.

The equation may be written in the form

$$\frac{\partial}{\partial v}\left(\frac{\partial T}{\partial u}+\frac{2u}{u^2+v^2+1}T\right)+\frac{2v}{u^2+v^2+1}\left(\frac{\partial T}{\partial v}+\frac{2u}{u^2+v^2+1}T\right)=0.$$

This equation may be viewed as an ordinary differential equation for determining the quantity in parenthesis; the arbitrary "constant" is then a function of u. Solving this equation, we have

$$\frac{\partial T}{\partial u}+\frac{2u}{u^2+v^2+1}T=\frac{\bar{\psi}(u)}{u^2+v^2+1};$$

or

$$(u^2+v^2+1)\frac{\partial T}{\partial u}+2uT=\bar{\psi}(u).$$

Hence

$$(u^2+v^2+1)T=\int\bar{\psi}(u)du+\varphi(v)=\psi(u)+\varphi(v).$$

Therefore, we have

$$T=\frac{\psi(u)+\varphi(v)}{u^2+v^2+1},$$

where $\psi(u)$ and $\varphi(v)$ are arbitrary functions of u and v respectively.

Solved in a different manner by the Proposer.

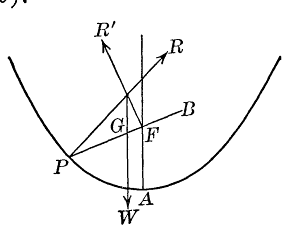
MECHANICS.

267. Proposed by G. H. LIGHT, Purdue University.

A parabolic curve is placed in a vertical plane with its axis vertical and vertex downwards, and inside it, and against a peg in the focus, and against the concave arc, a smooth uniform and heavy beam rests; required the position of equilibrium. [From Bowser's Mechanics, Ex. 37, p. 96.]

SOLUTION BY CHRISTIAN HORNUNG, Heidelberg University, Tiffin, O.

Let PB be the beam of length l, of weight W, resting on the peg at the focus, F; let $AF=p$, $AFP=\theta$, and $GP=l/2$. From the polar equation of the parabola we have $FP=2p/(1+\cos\theta)$.



Since all the surfaces are smooth the resistances, R and R', at P and F respectively, are normal and therefore the angle $BPR=\theta/2$.

Using the equations $\Sigma x=0$, $\Sigma y=0$ and Σ(moments about F) $=0$, which

express the conditions of equilibrium, we have respectively:

$$R \sin\frac{\theta}{2} - R' \cos\theta = 0, \qquad R\cos\frac{\theta}{2} + R' \sin\theta - W = 0,$$

$$R\left(\frac{2p}{1+\cos\theta}\right) - W\left(\frac{2p}{1+\cos\theta} - \frac{l}{2}\right) = 0.$$

From these three equations we get

$$\cos^4\frac{\theta}{2} = \frac{p}{l}, \qquad \cos\frac{\theta}{2} = \left(\frac{p}{l}\right)^{\frac{1}{4}}, \qquad \theta = 2\cos^{-1}\left(\frac{p}{l}\right)^{\frac{1}{4}},$$

which is the result given as the answer to the problem by the author.

Excellent solutions were received from M. E. Graber, E. B. Escott, J. E. Sanders, J. Scheffer, A. M. Harding, M. J. McCue, Mrs. H. E. Trefethen, S. G. Barton and the Proposer.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

193. Proposed by W. E. HEAL, Washington, D. C.

Develop a formula for the value of x in the equation $p^x + q^x = s^x$, p, q, and s being integral numbers.

Solution by B. F. Yanney, University of Wooster.

By letting $a = p/s$, $b = q/s$, the given equation may be reduced to the form

$$a^x + b^x - 1 = 0.$$

Setting $y = a^x + b^x - 1$, and remembering that

$$a^x = 1 + x\log_e a + \frac{x^2\log_e^2 a}{2\,!} + \frac{x^3\log_e^3 a}{3\,!} + \cdots,$$

we have

$$y = 1 + x\log_e(ab) + x^2\frac{\log_e^2 a + \log_e^2 b}{2\,!} + \cdots.$$

By reversion, we get

$$x = A(y-1) + B(y-1)^2 + C(y-1)^3 + D(y-1)^4 + \cdots,$$

where

$$A = \frac{1}{\log_e(ab)}, \qquad B = -\frac{\log_e^2 a + \log_e^2 b}{2!\log_e^3 a},$$

$$C = \frac{2\cdot 3!\,(\log_e^2 a + \log_e^2 b)^2 - (2!)^2\log_e(ab)\cdot(\log_e^3 a + \log_e^3 b)}{(2!)^2 3!\log_e^5 a},$$

$$D = -\{5\cdot 3!\,4!\,(\log_e^2 a + \log_e^2 b)^3 - 5\cdot(2!)^2 4!\log_e(ab)\cdot(\log_e^2 a + \log_e^2 b)(\log_e^3 a + \log_e^3 b) + (2!)^3\,3!\log_e^2(ab)\cdot(\log_e^4 a + \log_e^4 b)\} \div \{(2!)^3\,3!\,4!\log_e^7 a\},$$

etc.

Now, by hypothesis $y = 0$.

Hence

$$x = -A + B - C + D - \cdots.$$

Also solved by A. H. Holmes. Mr. Yanney sent a solution of 188 after the list of solutions was sent to the printer.

MISCELLANEOUS QUESTIONS.

EDITED BY R. D. CARMICHAEL.

QUESTION.

6. In what ways and to what extent will the teaching of mathematics and the content of the curriculum probably be affected by the increasing demand for vocational training?

REPLIES.

2. It is clear, on the most casual observation, that the average college curriculum in mathematics is unbalanced in two respects: (1) algebra predominates over geometry, (2) analytic geometry predominates over our synthetic geometry. This reacts in two undesirable ways: (1) to deprive the college student of a rich and interesting field of study, (2) to give a very one-sided training for prospective teachers of high school mathematics. What can be done to remedy this situation?

IV. Remarks by R. D. Carmichael, Indiana University.

The discussion of this question both by Professor Bussey in the issue of November, 1913, and by several correspondents in the January issue has been stimulating. There can be no doubt that our college curriculum is unbalanced in some important respects. In fact it would probably be hard to give a good reason why certain topics are retained in our courses and still harder to say why others have not been introduced.

It seems clear that the claim of synthetic geometry is a valid one. The beauty of the subject cannot be doubted, and its appeal to the majority of students will probably be granted. It adds to the esthetic delight of one's intellectual life when he is able to picture space as shot through with the beautiful relations which have been created by geometers. It gives him a language of precision and imagery in which to set forth the relations of phenomena in the world of external things through which he must get about in his daily experience.

The purpose of these remarks, however, is not primarily to emphasize these matters, although the present state of our curriculum shows that they are in need of emphasis; it is rather to point out what appears to be a wrong judgment concerning analysis or a wrong emphasis of its difficulties, as expressed in the following words in one of the communications referred to: "There comes a time in the life of every mathematical student when the continual grind of analysis becomes unbearable."

To the present writer analysis is not a "grind" and has never been; on the other hand, it throws over him a peculiar fascination. His deepest intellectual pleasure is the experience of beholding the beautiful threads of connection among these abstract creations of the mind and in witnessing the marvelous and unexpected ways in which they have contributed to our better understanding of the external world and its phenomena.

If this experience were confined to a single individual it should pass unmentioned. But there are not a few who feel the same sense of charm in the

presence of the far-reaching results of analysis, and find in them (directly or indirectly) the chief source of their pleasure in mathematical discipline; and this experience has been theirs through the whole course of their study.

It is doubtless true, however, that many students find analysis anything but pleasant and that this would continue to be their experience although they gave much attention to its cultivation. For these we need synthetic geometry in all its beauty; and not for these only, but for all students who come to us for mathematical training. Let us adjust our courses so that every college student may have the opportunity to go into this rich and interesting field of study.

3. In connection with the theory of the conduction of electricity through gases, one is led to the differential equation

$$y\frac{d^2y}{dx^2} + a\left(\frac{dy}{dx}\right)^2 + b\frac{dy}{dx} = cy + d = 0, \tag{1}$$

where a, b, c, d are constants. For unrestricted values of a, b, c, d the solution of this differential equation presents peculiar difficulties, the series solutions obtained by the customary methods having (apparently) too small a range of convergence to be satisfactory from the point of view of electrical theory. The general solution of this equation is wanted in case it can be found. If no general solution is obtained for unrestricted a, b, c, d, it is desirable to know special values of a, b, c, d or special relations among a, b, c, d which make it possible to find the general solution; and this solution is desired in each case.

I. Remarks by X.

A list of some cases in which the differential equation (1) may be solved by elementary means is the following:

1. When $b = c = d = 0,\ a \neq 0,\ -1$. The general solution in this case is readily found to be

$$y = (\alpha x + \beta)^{\frac{1}{a+1}},$$

where α and β are arbitrary constants.

2. When $b = c = d = 0,\ a = -1$. The general solution is

$$y = e^{\alpha x + \beta},$$

where α and β are arbitrary constants.

3. When $a = c = d = 0,\ b \neq 0$. It is easy to show that the general solution is given by the equation

$$\int_0^{\alpha y} \frac{dt}{\log t} + bx + \beta = 0,$$

where α and β are arbitrary constants.

4. When $a = b = d = 0$. The general solution is

$$y = -\frac{cx^2}{2} + \alpha x + \beta.$$

5. When $a = b = c = 0,\ d \neq 0$. The general solution is given by the equation

$$\int_0^y \frac{dt}{\sqrt{\alpha - 2d \log t}} + x + \beta = 0.$$

6. When $c = d = 0$, $a \neq 0$, $b \neq 0$. The general solution is given by the equation

$$\int_0^y \frac{at^a dt}{bt^a - \alpha} + x + \beta = 0.$$

7. When $a = b = 0$, $c \neq 0$, $d \neq 0$. The general solution is given by the equation

$$\int_0^y \frac{dt}{\sqrt{\alpha - 2ct - 2d \log t}} + x + \beta = 0.$$

Returning now to the general equation (1), put

$$t = \alpha x + \beta,$$

where α and β are any constants except that α is to be different from zero. Then the differential equation becomes

$$y\frac{d^2y}{dt^2} + a\left(\frac{dy}{dt}\right)^2 + \frac{b}{\alpha}\frac{dy}{dt} + \frac{c}{\alpha^2}y + \frac{d}{\alpha^2} = 0. \tag{2}$$

If the original equation (1) has the solution

$$y = f(x, a, b, c, d), \tag{3}$$

then equation (2) has the solution

$$y = f\left(t, a, \frac{b}{\alpha}, \frac{c}{\alpha^2}, \frac{d}{\alpha^2}\right),$$

and hence the original equation (1) has the solution

$$y = f\left(\alpha x + \beta, a, \frac{b}{\alpha}, \frac{c}{\alpha^2}, \frac{d}{\alpha^2}\right). \tag{4}$$

Therefore, *if any solution* (3) *of* (1) *is known, actually involving x, a solution exists in the form* (4) *involving two arbitrary constants.* This fact may be of use in obtaining the general solution of (1).

NOTES AND NEWS.

UNDER THE DIRECTION OF FLORIAN CAJORI.

Professor M. Fréchet, of Poitiers, France, is expected to lecture at the University of Illinois during the next academic year.

The Royal Society has awarded the Sylvester medal to J. W. L. Glaisher, F.R.S., for his mathematical researches.

Dr. Karl Boehm, of Heidelberg, succeeds Professor G. Faber as professor of mathematics in the University of Königsberg.

A portrait of Professor Horace Lamb, well known for his researches in mathematical physics, was presented by subscribers to the University of Manchester, where he has filled the chair of mathematics since 1885. The portrait was painted by his son, Henry Lamb, a rising young artist.

The German Mathematical Society recently subscribed 500 francs per year, for five years, in addition to its earlier subscription of 5,000 francs, toward the cost of the publication of Euler's Complete Works. This cost is now estimated at one million francs, which is twice the original estimate.

The Irish Journal of Education for December, 1913, contains the second instalment of the Report of the National Committee of Fifteen on Geometry Syllabus (U. S. A.), which it is reprinting in full. Five hundred copies of this report have recently been purchased by the State Department of New Jersey and placed in the hands of all teachers of geometry in the state.

Dr. Edward L. Dodd, of the University of Texas, contributes to the *Giornale di Matematiche di Battaglini* (Vol. 51, 1913) an interesting article on "A justification of empirical probability based upon an undetermined *a priori* probability." The topic is particularly important in life insurance.

A biography and popular account of the discoveries of T. J. J. See has been written by W. L. Webb. The book includes a statement in non-technical language of See's theories on the creation of the earth and the heavens.

Cosmological speculations receive attention in the January number of *Scientia,* an international review of scientific synthesis, edited by Italian scholars. Professor T. C. Chamberlin, of the University of Chicago, has an article on Planetesimal Hypotheses; the astronomer A. C. D. Crommelin, of Greenwich, writes on the Capture Theory of Satellites; Dr. T. J. J. See discusses the law of nature in celestial evolution.

Sir Robert S. Ball, Lowndean professor of astronomy and geometry in the University of Cambridge, died November 25, 1913, at the age of 73. Along the lines of applied mathematics he contributed a work on *Experimental Mechanics* in 1871, and one on the *Theory of Screws* in 1900, which embodies important researches begun by him in 1870 and published in a series of memoirs. After 1884 he did very little observing, probably on account of trouble with one of his eyes, accidentally injured in his youth and finally removed in 1897.

Dr. Artemas Martin, of Washington, D. C., has recently published in the *Mathematical Magazine* a paper on "The algebraic solution of the famous three point problem." The problem is to determine a point in the plane of a given triangle at which the sides subtend given angles. The two papers read by Dr. Martin at the International Congress of Mathematicians at Cambridge, England,

in 1912 have been published in the proceedings of the congress. The titles of these papers are "On rational right-angled triangles" and "On powers of numbers whose sum is the same power of some number."

Yoshio Mikami read before the Tôkyô Mathematico-Physical Society, and brought out in the *Tôkyô Sûgaku-Buturigakkwai Kizi* (2 S., Vol. VII, No. 9, p. 157), an article on the formula for an arc of a circle, found in the "Kwatsuyō Sampō," a treatise printed in 1712 as a posthumous work of Seki Kōwa and probably composed as early as 1683. Mikami gives a fuller historical and analytical discussion than previously given, of Seki Kōwa's formula,

$$1276900(d-s)^5a^2 = 5107600d^6s - 23835415d^5s^2 + 43470240d^4s^3 - 37997429d^3s^4 + 15047062d^2s^5 - 1501025ds^6 - 281290s^7 \cdots,$$

which may be used for the numerical calculation of an arc a of a circle of diameter d, whose altitude is s.

Dr. A. Mitzscherling has produced an interesting book, *Das Problem der Kreisteilung* (Teubner, 1913), which treats the subject historically. It considers the construction of regular polygons, the trisection and multisection of angles, the mechanisms by which these sections can be effected, and also the methods of approximation. Eight different elementary methods of inscribing a regular polygon of 17 sides are given in detail. For multisection 27 different curves are used. Among American writers reviewed in the book are Dexter, J. B. Miller, T. W. Nicholson and E. W. Hyde.

C. F. B. Funk has an article in Schotten's *Zeitschrift* (November, 1913, p. 463), in which he offers a detailed exposition of logarithmic theory along the lines suggested by Felix Klein in his *Elementarmathematik vom höheren Standpunkt*. It is well known that Klein exposes the defects of the current definition of logarithms and suggests that, for school purposes, logarithms be defined geometrically by means of the equilateral hyperbola. He leaves the elaboration of details of his program to the experienced teacher of elementary mathematics. A detailed exposition was given by C. Frenzel in Schotten's *Zeitschrift*, Vol. 44, p. 1, and is now attempted for a second time by Funk.

The question of the adoption of the metric system in the United States and Great Britain will soon become a matter not merely of educational and scientific concern, but also of economic and trade importance. As pointed out in *Nature* (Nov. 27, 1913, p. 384) the adoption of the metric system promises to become, in the near future, a necessity in our trade dealings with China, Japan and Siam. These countries have taken steps to establish that system. In Japan the system is now obligatory for the services of the customs, excepting a few articles, also for the army, for medicine and for electrical work. Russia is taking steps pointing toward the general introduction of the metric system. The rest of Continental Europe has already adopted the metric reform.

The definitive program for the meeting of the International Commission on

the Teaching of Mathematics to be held at Paris in April, 1914, will consist of three sessions daily from April 1 to 4 inclusive:

First day: Session of the Central Committee. Business Session of the Commission. Session of the Mathematical Society of France.

Second day: General opening session. President, L. Poincaré, Director of Secondary Instruction, representing the Minister of Public Instruction. Address of welcome by P. Appell, Dean of the Faculty of Science, member of the Institute. Response by the President of the commission, F. Klein, of Göttingen. Address by the representative of the Minister of Public Instruction. Lecture by E. Borel on "The adaptation of instruction to the progress of science." Lecture by M. D'Ocagne on "The role of mathematics in the engineering sciences." Working session, taking up question A:—"The introduction of the elementary notions of the differential and integral calculus into secondary instruction." General reporter, E. Beke of Budapest.

Third day: Working session, taking up question B:—"The mathematical instruction of engineering students." General Reporter, P. Staeckel, of Heidelberg. Working session. Discussion on the teaching of mathematics in engineering schools. Meeting of the Society of Civil Engineers.

Fourth day: Working session. Conclusion of the discussion of questions A and B. Summaries by the general reporters. Business session. Consideration of the future work of the commission, in particular of the program for the meeting of the commission to be held at Munich in 1915, whose principal topic has already been fixed as "The theoretic and practical preparation of instructors of mathematics for the various stages of work." Reception by Prince Bonaparte.

Mr. C. Bourlet, who died last August, has been succeeded in the Commission by C. Bloche, and Messrs. A. de Saint Germain; and C. A. Laisant, who have resigned from the Commission for reasons of health and age, have been replaced by J. Hadamard and M. d'Ocagne.

The sessions will be held at the Sorbonne except as otherwise specified. The general opening session will be public. Admission to the working sessions will be limited to members of the Commission and of the various National Subcommissions and Committees, and to such other persons as shall have been furnished with tickets of admission by the General Secretary.

The Philosophical Society of France, in conjunction with the publishers of the Encyclopedia of the Mathematical Sciences, invites the mathematicians present in Paris on the occasion of this Congress to a series of sessions to be held April 6–8, at which various questions on the philosophy of mathematics will be considered. The Physical Society of France will hold its annual session and exposition of recent apparatus at Paris, April 15–17.

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OPTICAL INTERPRETATIONS IN HIGHER GEODESY.[1]

By WM. H. ROEVER, Washington University.

It is the object of this paper to show that certain curves which are defined in higher geodesy may be given simple optical interpretations. In order to be able to state what these interpretations are it will be necessary to define certain terms in both higher geodesy and optics.

§ 1. **Terms in Higher Geodesy.** Let us assume a set of rectangular axes O–x, y, z which are at rest with respect to the solid part of the rotating earth and such that the axis Oz is coincident with the earth's axis of rotation, the positive direction of Oz being from the celestial south pole to the celestial north pole, and the positive directions of Ox and Oy being such that $+Oy$ may be obtained from $+Ox$ by rotating $+Ox$ through 90° in the direction in which the earth rotates. Let us denote by P_0, (x_0, y_0, z_0) a point (on or near the earth's surface) which is at rest with respect to these axes. The field of force which determines weight, that is, the field in which a plumb-line is in equilibrium, is at rest with respect to these axes. The straight line which passes through P_0 and gives the direction of the force of this field at P_0, is defined as the *vertical at* P_0. The vertical at P_0 coincides with the string of a plumb-line,[2] the plumb-bob of which is situated at P_0. The *astronomical latitude at* P_0 is the complement ϕ of the angle which the vertical at P_0 (in the direction of the zenith) makes with the earth's axis (in the direction of the celestial north pole). The vertical at a general point P_0 does not intersect the earth's axis of rotation.

The *astronomical meridian plane at* P_0 is the plane which passes through the vertical at P_0 and is parallel to the earth's axis of rotation. The *astronomical longitude at* P_0 is the angle λ which the meridian plane at P_0 makes with a fixed plane through the axis of rotation (we here assume that the fixed plane is the plane zOx) measured from 0° to 360° in the direction in which the earth rotates.

[1] Presented to the American Mathematical Society (Chicago Section), April 5, 1912.

[2] It is assumed that the string of the plumb-line is weightless and perfectly flexible, and that the plumb-bob is a heavy particle.

The *horizontal plane at* P_0 is the plane which passes through P_0 and is perpendicular to the vertical at P_0. The *north-and-south line at* P_0 is the line of intersection of the meridian and horizontal planes at P_0. The *east-and-west line at* P_0 is the

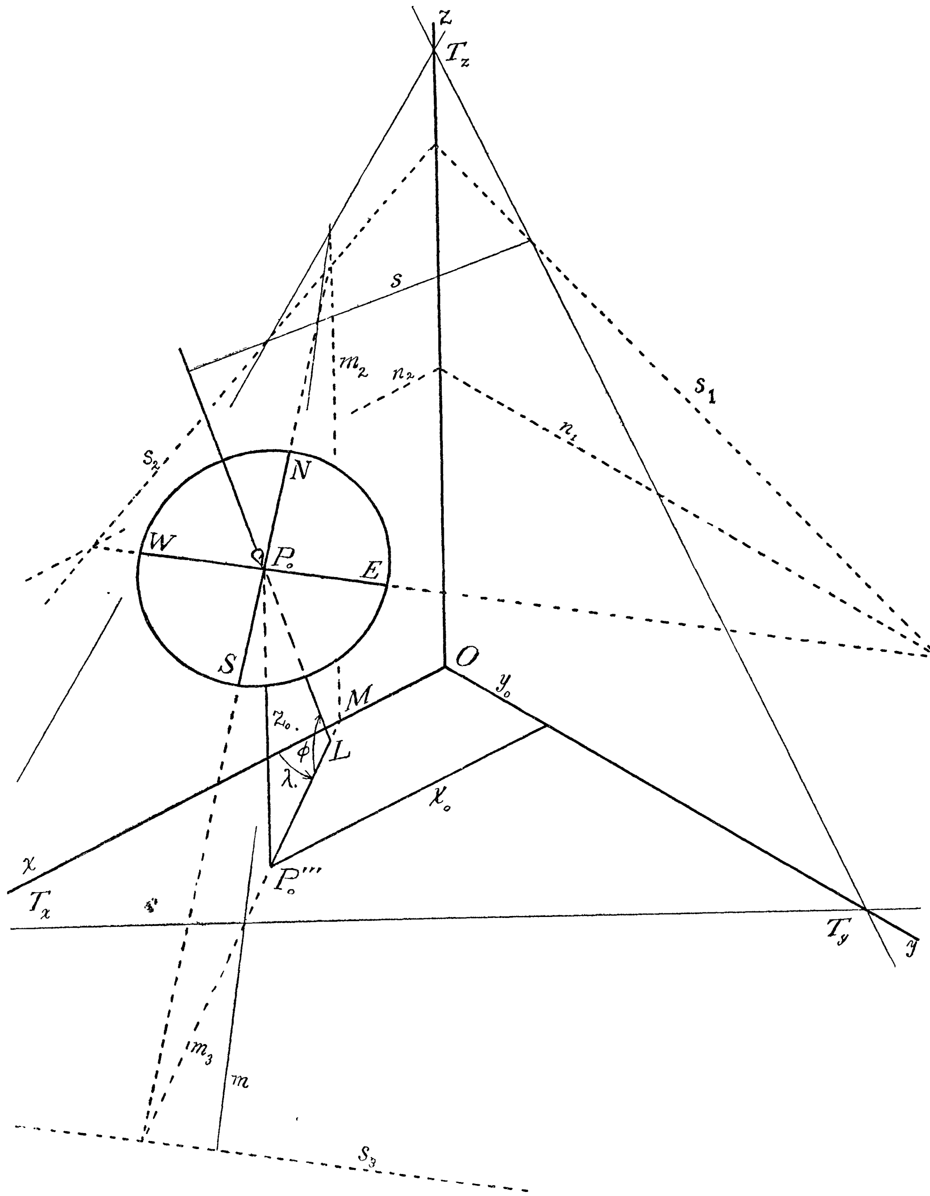

FIG. 1.

line which passes through P_0 and is perpendicular to the meridian plane at P_0. It is evident that this line is the intersection of the horizontal plane at P_0 by the plane which passes through P_0 and is perpendicular to the earth's axis of rotation.

Thus in Fig. 1, in which the system of axes O–x, y, z is represented in orthographic projection,[1] is shown the point P_0, whose projection on the plane xOy is P_0''', the vertical P_0L, whose projection on the plane xOy is $P_0'''L$, the latitude $\phi = \measuredangle P_0'''LP_0$, the meridian plane LP_0P_0''' (also shown by its traces m_2, m_3), the longitude $\lambda = \measuredangle T_xMP_0'''$, the horizontal plane $NESW$ (also shown by its traces s_1, s_2, s_3), the north-and-south line NP_0S (coincident with the line connecting the intersection of s_2 and m_2 with that of s_3 and m_3), the east-and-west line EP_0W (coincident with the line connecting the intersections n_1s_1 and n_2s_2, where n_1 and n_2 are traces of the plane which passes through P_0 and is parallel to xOy).

A level surface is an equipotential surface of the field of force referred to above, *i. e.*, a surface, fixed with respect to the axes O–x, y, z, at each point of which the normal is the vertical. A *curve of constant astronomical latitude* is a locus of points, on a level surface, for which ϕ is constant. In particular, *the equator* is the locus for which $\phi = 0$. A *curve of constant astronomical longitude* is a locus of points, on a level surface, for which λ is constant.[2] The east-and-west line at a general point P_0 is *not* tangent to the curve of constant latitude which passes through P_0. A curve at each point of which the tangent is the east-and-west line we shall call an *east-and-west curve*.[3] It is not difficult to see that the east-and-west curves lie on level surfaces and also in planes which are perpendicular to the earth's axis of rotation.[4] Hence the one-parameter family of east-and-west curves, which lie on a particular level surface, are cut from this level surface by the one-parameter family of planes which are perpendicular to the earth's axis of rotation.

§ 2. **Terms in Optics.** If rays of light meet the surface of a body, the surface is said to be illuminated in the region where the rays meet it. The appearance of brightness of an illuminated surface depends upon the quantity of light which this surface sends to the retina of the eye. The quantity of light which an element of surface sends to the eye depends on the *intensity of illumination* of this surface element and on the reflecting power, or *albedo*, of the surface. For parallel ray illumination (such as that due to the sun or other distant sources of light) the *intensity of illumination* of a plane surface element, whose normal makes an angle ϵ with the direction of the rays of light, is given by the formula

$$B = L' \cos \epsilon,$$

where L' is the intensity of illumination of a plane surface which cuts the rays of light orthogonally.

Of the quantity of light which falls upon the surface of a body, a portion is

[1] This figure is correctly drawn according to the rules of axonometry. $T_xT_yT_z$ is the axonometric triangle, m and s are the axonometric traces of the planes $[m_2, m_3]$ and $[s_2, s_3]$ respectively. Therefore in the plane of the drawing the line s is perpendicular to the line P_0L and the line m is perpendicular to the line s_3.

[2] The above definitions are practically the same as those given by Pezzetti, *Trattato di Geodesia Teoretica* (1905), § 5.

[3] Similarly, *north-and-south curves* might be defined. In general, such curves are different from the curves of constant longitude.

[4] This is a consequence of the fact, stated above, that the east-and-west line at P_0 is the intersection of the horizontal plane at P_0 by the plane which passes through P_0 and is perpendicular to the earth's axis. For, the horizontal plane at P_0 is the tangent plane at P_0 to the level surface which passes through P_0, and the plane through P_0 and perpendicular to the axis, is its own tangent plane, and in general the tangent to the curve of intersection of two surfaces is the intersection of the tangent planes to the two surfaces.

absorbed, another portion is reflected and still another portion may be transmitted through the body. Only that portion which is reflected contributes to the brightness. The manner of the reflection depends upon the nature of the surface. A reflecting surface is said to be *perfectly smooth* when corresponding to each incident ray there is but a single reflected ray, the reflection occurring according to the law that the angles of incidence and reflection are equal and lie in the same plane: that is, *the normal to the surface at the point of reflection bisects the angle formed by the incident and reflected rays.* A reflecting surface is said to be *perfectly rough or dull* when it scatters the incident light which illuminates it in all directions and appears equally bright from all directions of view. The *brightness* of a surface element f is defined as the quantity of light which this surface element sends to the retina of the eye, divided by the area of the retinal image of f. For parallel ray illumination the *brightness* of a surface element is given by the formula

$$H = AL' \cos \epsilon,$$

where A is the albedo of the surface and L' and ϵ are quantities already defined. The brightness is independent of the direction from which the surface is viewed, provided the surface is perfectly dull.

For a surface which is not plane, the quantity H varies from point to point, but has constant values along certain curves of the surface. These curves, provided L' and A are constants, are the loci of points of the surface for which the angle ϵ, between the surface normal and the direction of the rays of light, has constant values. They are called *lines of constant intensity of illumination or curves of constant brightness.* The particular curve of this kind for which $\epsilon = 90°$, $\cos \epsilon = 0$, is called *the line of shade.*

Now let us consider a perfectly smooth reflecting surface as defined above. Such a surface has no brightness at any of its points, except at those isolated points from which emanate those unique rays of reflection which pass to the observer's eye. These points are very brilliant and hence are called *brilliant points.*[1]

Loci of Brilliant Points. The two ideal types of reflecting surface which are defined above may be regarded as extremes between which there lies another type which is worthy of consideration. For the perfectly smooth reflecting surface there corresponds to each incident ray but a *single reflected ray,* which emanates from the point where the incident ray pierces the surface, while for the perfectly dull reflecting surface there corresponds to each incident ray the *two-parameter family of reflected rays* which emanate from the point where the incident ray pierces the surface. For a type of reflecting surface which has already been considered by the author,[2] there corresponds to each incident ray the *one-parameter family of reflected rays* which emanate from the point P where the

[1] The above definitions and formulas are those given by Weiner, *Lehrbuch der Darstellenden Geometrie,* in the chapters on Beleuchtungslehre, Vol. I, p. 390, Vol. II, p. 200.

[2] See the introduction to the author's paper, "Brilliant Points of Curves and Surfaces," *Trans. of the American Math. Society,* Vol. 9, No. 2, pp. 245–279.

incident ray pierces the surface and form a cone of revolution whose vertex is P and whose axis is a tangent line to the surface at the point P. Such a surface may be realized[1] by scratching a perfectly smooth reflecting surface along the members of a one-parameter family of curves of the surface. The surface of a piece of metal can be thus scratched by holding a piece of emery cloth against

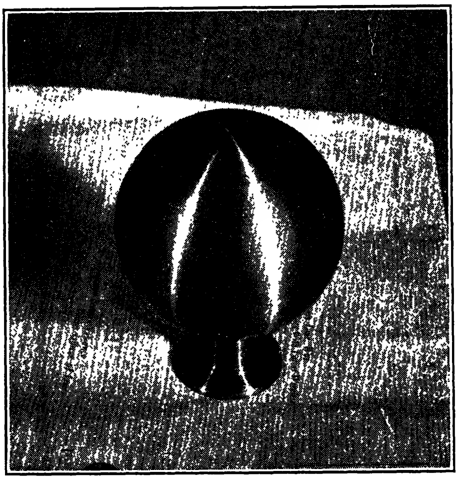

FIG. 2.

it while it rotates in a lathe. Whereas perfectly smooth reflecting surfaces, which are illuminated, exhibit isolated *bright points* (defined above as brilliant points), surfaces of this type exhibit *bright curves*. See the photograph, Fig. 2. These curves are the loci of points which the author has defined as *brilliant points of curves*.[2]

According to the law of reflection stated on page 72 and to the suggestion made in the last footnote, we have the following

Definitions. A point P is said to be a brilliant point of the surface t (or of the curve c, plane or twisted) with respect to the source (of light) P_1 and the recipient (an observer's eye) P_2 when the internal bisector of the angle P_1PP_2 is the normal to the surface t (or a normal to the curve c) at the point P. In

[1] Of course, only approximately, as are also the other two types.

[2] In order to get the concept of " brilliant point of a curve " and to be able to see that there corresponds to each incident ray a cone of reflected rays, let us think of the scratches as forming ridges and furrows which are perfectly smooth, and furthermore let us think of each ridge or furrow as belonging to a tube-surface. As the cross-section of a tube-surface approaches zero, a brilliant point of the surface approaches a point of the curve which is approached by the vanishing tube-surface. This consideration suggests the definition of brilliant point of a curve which is here given. Those rays, reflected from a perfectly smooth tube-surface, which correspond to incident rays which lie in a plane, form a ruled surface. As the cross-section of the tube-surface approaches zero, the narrow band of incident rays approaches a single ray, and the ruled surface of reflected rays approaches one nappe of a cone of revolution, of which the vertex is the point Q where the incident ray meets the curve approached by the vanishing tube-surface, the axis is the tangent at Q to this limit curve and the angle is the same as that which the incident ray makes with the axis of this cone extended in the direction opposite to that on which this nappe lies.

particular P_1 and P_2 may be infinitely distant. As a consequence of these definitions the following theorem is immediately evident.

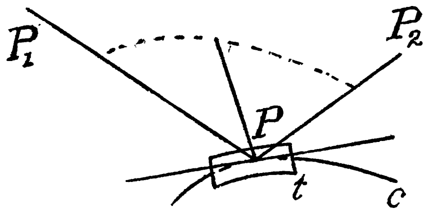

FIG. 3.

THEOREM. *The brilliant points of a surface (or a curve) are those of its points at which it is tangent to ellipsoids of revolution of which P_1 and P_2 are the foci. If, in particular, P_1 and P_2 are infinitely distant the confocal ellipsoids are replaced by parallel planes.*

If we denote by l_1, m_1, n_1 and l_2, m_2, n_2, the direction cosines of the directed lines PP_1 and PP_2 respectively, and by l, m, n numbers proportional to the direction cosines of the normal to the surface t (or the tangent line to the curve c) at the point P, then the condition that P should be a brilliant point, with respect to P_1 and P_2, of the surface is

$$(1) \qquad (l_1 - l_2)l + (m_1 - m_2)m + (n_1 - n_2)n = 0, \qquad \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0;$$

and of the curve c is[1]

$$(2) \qquad (l_1 + l_2)l + (m_1 + m_2)m + (n_1 + n_2)n = 0.$$

§ 3. **The Analogies.** Now that we have defined all of the necessary terms, we are able to state what the interpretations really are to which reference was made above. In order to do this let us think of a level surface of the earth's weight-field of force as being (1) a perfectly dull reflecting surface, (2) a reflecting surface of the type last defined in § 2 and scratched along the members of its one-parameter family of east-and-west curves.

We then have the following theorems:

THEOREM I. *The curves of constant astronomical latitude on a level surface are also the curves of constant brightness of this surface, regarded as a perfectly dull reflecting surface, when the source of illumination is an infinitely distant light in the direction of the earth's axis of rotation. In particular, the equator is the line of shade.*

THEOREM II. *The curves of constant astronomical longitude on a level surface are also the loci of the brilliant points of the one-parameter family of east-and-west curves of this surface, with respect to infinitely distant sources and recipients.*[2]

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[1] For more details see the author's paper already referred to.

[2] That is, if a level surface of the earth, supposed metallic, were scratched along its east-and-west curves, an observer on the moon, or any other distant body in a general direction, would see as a curve of light some curve of constant astronomical longitude, the curve being the assemblage of the images of the sun, which may be in any general direction, in the reflecting scratches.

The truth of theorem I follows immediately from the definitions of "brightness" and "astronomical latitude." The first of these quantities is proportional to cos ϵ and the second, ϕ, is equal to $90° - \epsilon$, since the rays of illumination are parallel to the earth's axis. Hence the locus of points on a level surface for which ϵ is constant, is in one case a curve of constant brightness and in the other, a curve of constant astronomical latitude.

Theorem II may be proved geometrically by means of the special case of the theorem stated in § 2. According to this special case, a point P of a curve c is a brilliant point of c with respect to the infinitely distant points P_1 and P_2, if c is tangent at P to one of the single infinity of planes which are perpendicular to the line b which bisects internally the angle P_1PP_2. If, in particular, the curve c is an east-and-west curve of a level surface, it lies in a plane perpendicular to the earth's axis of rotation a. Therefore, in order to find a brilliant point of an east-and-west curve c, it is only necessary to determine a point P of c at which the tangent is parallel to the line d in which the plane of c is cut by a plane perpendicular to the line b. The plane μ which is normal to an east-and-west curve c at a point P of c, and therefore also parallel to the earth's axis a, is the astronomical meridian plane of P. In the plane of each of the single infinity of east-and-west curves of a level surface there is a line d, a point P at which the tangent to the east-and-west curve c which lies in this plane is parallel to d, and a corresponding plane μ. All of the lines d are parallel (being perpendicular to both a and b) and hence all of the planes μ are parallel and consequently all of the points P have the same astronomical longitude. Therefore the locus of the brilliant points P is a curve of constant astronomical longitude.

In order to get an analytic proof of Theorem II, let us represent by ω the angular velocity of the earth's rotation, and by U the function $f(x, y, z)$ which is defined by the integral

$$U = \int_V \frac{dm}{\rho}, \qquad \rho = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2},$$

where dm represents that element of mass of the earth which is situated at the point whose coördinates, with respect to the axes O–x, y, z, are x_i, y_i, z_i, the integration being extended throughout the whole volume of the earth. Then the potential function of the earth's weight-field of force is[1]

$$W = U + \frac{\omega^2}{2}(x^2 + y^2).$$

The equations of the level surfaces are of the form

$$W(x, y, z) = C, \tag{1}$$

where C has constant values. For the particular level surface which passes

[1] See Pezzetti, *loc. cit.*, § 2.

through the point P_0, (x_0, y_0, z_0), $C = W(x_0, y_0, z_0)$. The direction cosines of the vertical at P_0 are proportional to[1]

$$W_x{}^\circ, \quad W_y{}^\circ, \quad W_z{}^\circ.$$

Since the direction cosines of the earth's axis (here coincident with Oz) are

$$0, \quad 0, \quad \pm 1,$$

the equation of the astronomical meridian plane at P_0 is

$$W_y{}^\circ(x - x_0) - W_x{}^\circ(y - y_0) = 0, \tag{2}$$

and hence the astronomical longitude λ at P_0 is given by the relation

$$\tan \lambda = \frac{W_y{}^\circ}{W_x{}^\circ}, \tag{3}$$

and the differential equations of the two-parameter family of east-and-west curves are[2]

$$\frac{dx}{W_y} = \frac{dy}{-W_x} = \frac{dz}{0}. \tag{4}$$

Since in our particular problem the source P_1 and the recipient P_2 are infinitely distant, the quantities l_1, m_1, n_1, l_2, m_2, n_2 are constants, as are also the quantities

$$\alpha = l_1 + l_2, \quad \beta = m_1 + m_2, \quad \gamma = n_1 + n_2,$$

which are proportional to the directional cosines of the bisector b defined in the geometric proof of theorem II. If now we replace l, m, n by W_y, $-W_x$, O respectively, condition (2), § 2, takes the form

$$\alpha \cdot W_y - \beta \cdot W_x = 0 \quad \text{or} \quad \frac{W_y}{W_x} = \frac{\beta}{\alpha}. \tag{5}$$

This equation is the locus of the brilliant points, with respect to the infinitely distant points P_1 and P_2, of the two-parameter family of east-and-west curves (4). In its second form we recognize it also (by virtue of relation (3)) as the locus of points which have the same constant astronomical longitude $\lambda = \tan^{-1} (\beta/\alpha)$. Therefore the curve of intersection of this locus (5) with a level surface is both *a curve of constant astronomical longitude* on this level surface and *a locus of brilliant points*, with respect to the infinitely distant points P_1 and P_2, *of the one-parameter family of east-and-west curves* on this level surface. It is evident

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[1] The symbols W_x, W_y, W_z are used for the partial derivatives $\partial W/\partial x$, $\partial W/\partial y$, $\partial W/\partial z$ respectively, and the symbols $W_x{}^\circ$, $W_y{}^\circ$, $W_z{}^\circ$ stand for the values which these derivatives have at the point $x = x_0$, $y = y_0$, $z = z_0$.

[2] These differential equations may be written in the form: $W_x\,dx + W_y\,dy + W_z\,dz = 0$, $dz = 0$, from which it is evident that the solution is $W = C$, $z = k$, where C and k are constants. From this we see (what has already been stated in § 1) that the east-and-west curves are the intersections of the level surfaces with the planes perpendicular to the earth's axis.

that we can choose the directions of the infinitely distant points P_1 and P_2 (in an infinite variety of ways) so as to make the expression $\lambda = \tan^{-1}(\beta/\alpha)$ for the astronomical longitude have any value we wish.[1]

From the form of the function W, it is evident that the level surfaces (1) are, in general, *not* surfaces of revolution. If, in particular, the distribution of the earth's matter were such that the function U assumed the form $f(v, z)$, where $v = \sqrt{x^2 + y^2}$, then the level surfaces would be surfaces of revolution, the east-and-west curves would be identical with the curves of constant astronomical latitude, and the curves of constant astronomical longitude would be the meridian curves[2] of these surfaces of revolution.

Hence Theorem II yields the following corollary.

COROLLARY. *The locus of the brilliant points, with respect to an infinitely distant source P_1 and an infinitely distant recipient P_2, of the one-parameter family of curves which are cut from a surface of revolution by planes perpendicular to the axis, is a meridian curve of this surface of revolution.*[3]

The photograph on page 73 shows two such curves (due to two sources of light) on a brass sphere which has been properly scratched with emery cloth while rotating in a lathe. In reality, the sources and the recipient (observer's eye) are not infinitely distant, but they are practically so.

A THEOREM IN THE MODERN PLANE GEOMETRY OF THE ABRIDGED NOTATION.

BY ROBERT E. BRUCE, Boston University.

NOTE BY THE EDITORS. The following paper is another contribution fulfilling the spirit of the editorial in the January issue. While the subject matter is strictly elementary, the methods are elegant and forceful and should furnish an inspirational study for those interested in modern geometry. Those who may wish to read an introduction to the abridged notation will find chapter IV of Salmon's *Conic Sections* helpful.

Introduction. In common with polar reciprocation, projection, and certain other methods of modern geometry, the abridged notation may be used either for the proof of a theorem presupposed to be true or for the discovery of new theorems the exact form of which is unsuspected until the proof is complete. For either purpose an identity in terms of the abridged notation is selected and any number of algebraic operations performed upon it leading to a final identity. The first identity corresponds to the hypothesis, the last to the conclusion, and the intervening transformations to the proof. If the form of the theorem is

[1] It is also evident that the locus of the billiant points, with respect to a source P_1 and a recipient P_2, of any one-parameter family of curves on a surface σ, passes through the brilliant points, with respect to P_1 and P_2, of σ.

[2] A meridian curve of a surface of revolution is a curve which is cut from this surface by a plane which passes through the axis of rotation.

[3] If a surface of revolution were scratched along its meridian curves, the locus of the brilliant points would be a more complex curve. Such a locus has been considered by the author for the case of a sphere, in Vol XX, No. 10, p. 299 of this MONTHLY.

already suspected the identity of the hypothesis and the transformations are, as a matter of course, so selected as to bring the desired result. If, on the other hand, one desires to discover some theorem for the satisfaction of the discovery rather than for the theorem itself, the identity of hypothesis and the transformations may be selected almost at random; the work when done being interpreted in whatever geometry one chooses. In the following, the main theorem illustrates this latter use of the notation while the lemma proved in connection with corollary 8 illustrates the former.

Proof of Main Theorem. Let the k's be constants other than zero and let the S's be expressions of the same degree in the variables.

If

$$k_1S_1 + k_2S_2 \equiv k_3S_3 + k_4S_4$$

then

$$k_1S_1 - k_3S_3 \equiv k_4S_4 - k_2S_2$$

and

$$k_1S_1 - k_4S_4 \equiv k_3S_3 - k_2S_2.$$

Limiting the interpretation for simplicity to the case in point geometry where there are but two variables and where the S's are of the second degree we have the following:

If a conic through the intersections of conics S_1 and S_2 passes through the intersections of conics S_3 and S_4, then there is a conic through the intersections of S_1 and S_3 that passes through the intersections of S_4 and S_2. Similarly for S_1, S_4 and S_3, S_2. Hence the following:

Theorem. *If it is possible to divide four given conics into two groups of two each in such a way that the eight points of intersection within the groups are on a conic, then there will be a conic through the eight points of intersection in each of the other arrangements of the conics into two groups of two each.*

It is obvious that various other theorems may be derived from the proof by changing the degree of S, the number of variables, or the particular geometry used.

The following are some of the corollaries of the theorem as stated:

Corollary 1. Consider three intersecting conics with their nine pairs of opposite chords of intersection. Denote the conics by S_1, S_2, and S_3 respectively. Denote the pairs of opposite chords of intersection as follows: One pair for S_1 and S_2 by C_{12}', a second pair by C_{12}'', and the third by C_{12}'''. So for S_1, S_2 and S_2, S_3. Any one of these pairs of chords is a degenerate conic. Consider the four conics S_1, C_{13}', S_2, and C_{23}' grouping them as follows: (S_1, C_{13}'); (S_2, C_{23}'). The eight intersections within the groups are all on the conic S_3. Hence there exist two eight-point[1] conics,—one through the intersections of S_1 with S_2 and of C_{13}' with C_{23}', and the other through the intersections of S_1 with C_{23}', and of S_2 with C_{13}'. But the group[2] (S_1, C_{13}') may, aside from the above, be associated severally

[1] For sake of brevity the phrase "eight-point conics" will be used to denote the conics involved in the conclusion of the main theorem.

[2] This word is of course used in a non-technical sense.

with the groups (S_2, C_{23}'') and (S_2, C_{23}'''), the hypothesis of the theorem being fulfilled in each case. Hence for the group (S_1, C_{13}') there exist six eight-point conics derived through its association with S_2 and the various pairs of opposite chords of intersection of S_2 and S_3. But the group (S_1, C_{13}') may also be associated with the group (C_{23}', C_{23}''), since these two degenerate conics intersect on S_3, thus giving two more eight-point conics. Similarly for the groups (C_{23}', C_{23}''') and (C_{23}'', C_{23}'''). Hence there exist 12 eight-point conics for the various groupings above, using (S_1, C_{13}') as one pair of conics. But in place of (S_1, C_{13}') there may be substituted any one of the following pairs of conics: (S_1, C_{13}''), (S_1, C_{13}'''), (C_{13}', C_{13}''), (C_{13}', C_{13}'''), (C_{13}'', C_{13}'''). Each substitution gives, of course, 12 new eight-point conics by taking severally the various groups we have used above with (S_1, C_{13}'). Hence there exist 72 eight-point conics for the cases considered. But just as S_1 and S_2 with the chords used intersect upon S_3, so also do S_1 and S_3 with their chords intersect upon S_2, and S_2 and S_3 upon S_1. Hence there exist 216 eight-point conics through the various intersections. The above may be summarized as follows:

Given any three intersecting conics with their chords of intersection; there exist 216 *different conics each containing eight of the points of intersection of the conics and chords with one another.*

This corollary has been worked out in detail for the sake of clearness. A moment's consideration, however, makes it plain that the result may be found at once from the following product: $2 \times {}_4C_2 \times {}_4C_2 \times 3 = 216$.[1] It is also evident that the same method may be applied to the cases where the hypothesis gives four or more intersecting conics with their chords of intersection.

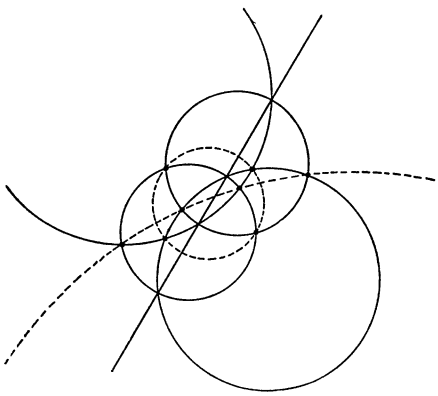


FIG. 1.

Corollary 2. If two pairs of conics have segments of the same two straight lines as opposite chords of intersection, the sixteen other points of intersection of the conics with each other are in two groups of eight each on two additional conics.

The two chords form a degenerate conic. And by hypothesis the four conics

[1] Where ${}_nC_r$ is the number of combinations of n things taken r at a time.

intersect upon it in two groups of two each. The conclusion, then, follows at once from the main theorem. The next two corollaries are special cases of the above.

Corollary 3. If two pairs of circles have different segments of the same line as their common chords, then the eight other points of intersection of the circles with each other are in groups of four each on two other circles. (See Fig. 1.)

The line at infinity is the second chord satisfying the hypothesis of corollary 2. The missing intersections are the circular points at infinity. Hence the conics of the conclusion are circles.

Corollary 4. Consider two quadrilaterals with different segments of the same two lines as diagonals. (See Fig. 2.) The pairs of opposite sides of the quadrilaterals are degenerate conics. Thus the hypothesis of corollary 2 is fulfilled. Hence there are two eight-point conics through the sixteen points of intersection of the sides of one quadrilateral with the sides of the other. This corollary is the basis of the construction of the following corollary.

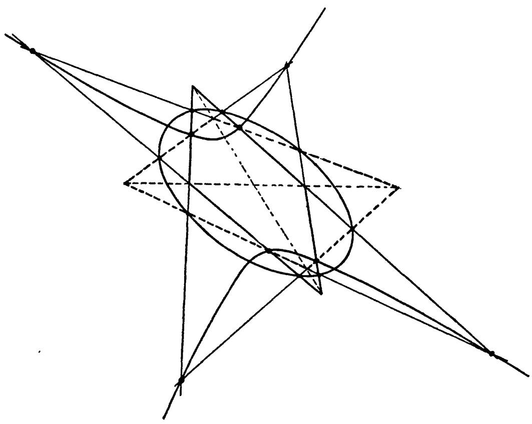


FIG. 2.

Corollary 5. *Construction of the conic through five given points.*[1]

Number the points 1, 2, 3, 4, 5. Let line 24 be α', 35 be β', 25 be b', and 34 be a'. Draw any two convenient lines, a and β, through 1.

Let Y be the intersection of a and a', W of a and b', I of β and α', and III of β and β'. Let the line WI be d, and $Y\,III$ be d'.

Let X be the intersection of d and a', II of d and β', IV of d' and α', and Z of d' and b'.

Let line XZ be b, and $II\,IV$ be α.

Let 6 be the intersection of b and β, 7 of a and α, 8 of b and α.

The points 6, 7, 8 are on the conic through 1, 2, 3, 4, 5. Other sets of points may be found by varying the arbitrary lines a and β.

[1] No diagram is drawn because the construction will be better appreciated by the reader if he draws the figure step by step. A figure is given in connection with a briefer statement of the construction added below.

Proof of the construction. The conic $\alpha\beta$ intersects the conic $\alpha'\beta'$ on the conic dd'. The conic ab intersects the conic $a'b'$ on the conic dd'. Hence the intersections of ab with $\alpha\beta$ and of $a'b'$ with $\alpha'\beta'$ are on a conic. But these points of intersection are the points 1, 2, 3, 4, 5, 6, 7, 8. Hence 6, 7, and 8 are on the conic through 1, 2, 3, 4, 5.

The following briefer statement of the method of construction has been found to satisfy the proof in all cases tested. (See Fig. 3.)

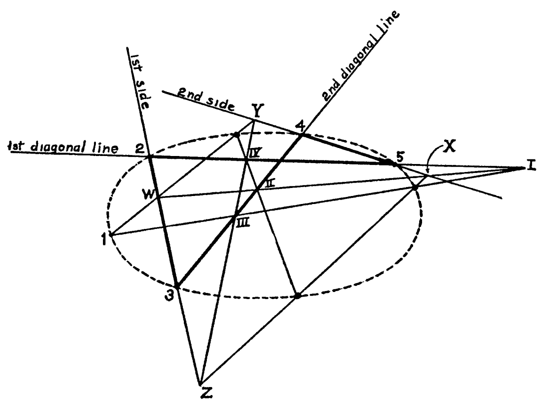


FIG. 3.

Selecting four of the given points, draw either pair of opposite sides and the two diagonals of the quadrilateral[1] of which these points are the vertices. Draw any two convenient lines through the fifth point.[2] Let the first of these lines intersect the first[3] and second diagonals in points I and III respectively. And let the second line intersect the first and second sides in points W and Y respectively. Call the points where the line $Y\,III$ meets the first side and the first diagonal Z and IV respectively; and the points where the line $W\,I$ meets the second side and the second diagonal X and II respectively. The following intersections are on the conic: $I\,III$ with $X\,Z$, $X\,Z$ with $II\,IV$, $II\,IV$ with $Y\,W$. By varying the lines through the fifth point as many points as desired may be found.

Corollary 6. *Pascal's Theorem:* "The intersections of the opposite sides of a hexagon inscribed in a conic are collinear." (See Figure 4.)

Let $ABCDEF$ be the given hexagon. Draw $C'F'$ any line that meets the conic, and form the two inscribed quadrilaterals $ABCF$ and $C'DEF'$. The degenerate conic formed by the lines BC and AF intersects the conic of the lines AB and CF on the given conic. Similarly for conics $C'D$, EF' and $C'F'$, ED. Hence these four conics obey the hypothesis of the main theorem and by pairing

[1] That is, the ordinary convex quadrilateral of the elementary euclidean geometry,—not the complete quadrilateral. If no three of the points are in the same straight line a selection of this character is always possible.

[2] In the figure this point is 1.

[3] The numbering throughout is arbitrary.

the conics as follows: $BC \cdot AF$ with $EF' \cdot C'D$ and $AB \cdot CF$ with $C'F' \cdot ED$ the points 1, 2, 3, 4, 5, 6, 7, 8 are seen to be on a conic. But if the line $C'F'$ coincides with the line CF, the points 6, 7, 1, 2, 5 are collinear on the line CF. Hence the

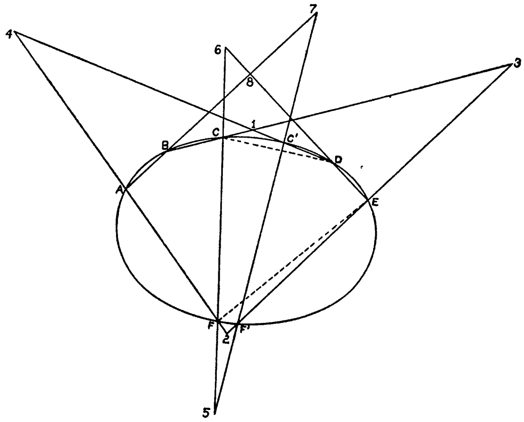


FIG. 4.

eight-point conic is degenerate and the points 4, 8, 3 are collinear on a line other than CF. But these points are the intersections of the opposite sides of the hexagon.

Corollary 7. "If three conics have a common chord the opposite common chords are concurrent." (*Salmon's Conic Sections*, Art. 266.) See Figure 5.

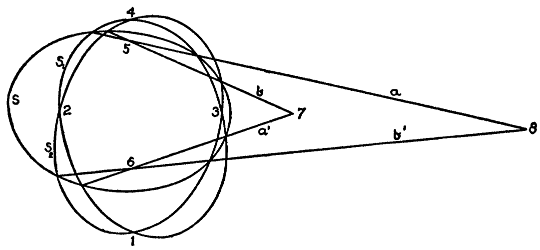


FIG. 5.

If the chords a and a' are associated with S_1, and b and b' with S_2, the two pairs of conics thus obtained obey the hypothesis of the main theorem, since the intersections within the groups are on S. Hence by associating chords $a \cdot a'$ with chords $b \cdot b'$ and conic S_1 with conic S_2 the points 1, 2, 3, 4, 5, 6, 7, 8 are seen to be on a conic. If b coincides with a the three conics have a common chord and the points 2, 3, 5, 7, 8 are collinear upon it. Hence the eight-point conic is degenerate and the points 4, 6, 1 are collinear. 6 is thus the point of concurrency of the three opposite chords of intersection. It is obvious that the proof may be applied to any three conics having a common chord.

Corollary 8. "If two conics have double contact with a third, their chords of contact with the third and a pair of their chords of intersection with each other are concurrent." (*Salmon's Conic Sections*, Art. 263.)

For the proof of this and other corollaries the following lemma is used in addition to the main theorem.

Lemma. If a conic intersects itself or is tangent to itself, it is degenerate and consists of two straight lines through the points of intersection or tangency.

If $S = 0$ is the equation of a conic, and $\alpha = 0$ and $\beta = 0$ the equations of straight lines, and k a constant other than zero; then $S + k\alpha\beta \doteq 0$ may be used to represent any conic intersecting S or tangent to S, the particular relationship depending on the relationship of α and β to S. (*Salmon's Conic Sections,* Arts. 249, 251.)

Since by hypothesis S is to intersect itself or to be tangent to itself, $S \equiv m(S + k\alpha\beta)$, where m is a constant other than zero.

Hence

$$S \equiv \frac{mk}{1 - m}\alpha\beta.$$

Now $m \neq 1$ since otherwise, from the first identity, the product $\alpha\beta$ would be identically zero. Hence the locus S is the same as the straight lines α and β combined.

Proof of corollary 8. Let S_1 have double contact with S and let S_2 be tangent to S at G and intersect it at H and K. The following groups of conics fulfill the hypothesis of the main theorem: $(S_1, EF \cdot EF)$ and $(S_2, GH \cdot GK)$.

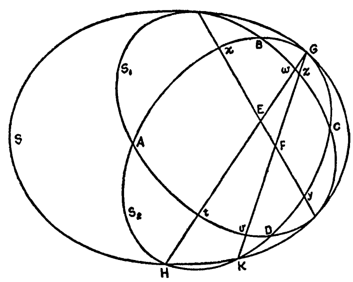


FIG. 6.

Hence there is a conic through the points $ABCD$ which is tangent to the line HG at E and to the line KG at F. But if F and E coincide, this conic is tangent to itself. Hence it is degenerate consisting of the two lines AC and BD passing through the point EF. Moreover, under these conditions the conic S_2 has double contact with S. Apparently any conic having double contact with S may be derived in this way and hence the corollary is proved.

Corollary 9. The second eight-point conic in figure 6 passes through the points t, v, w, z and has double contact with S_2 at x and y. Moreover, if S_2 has double contact with S, w coincides with z and t coincides with v, and this second eight-point conic has double contact with S_1 also. Hence the following: If two conics have double contact with a third conic they also have double contact with a fourth conic, the chords of contact being different segments of the same two lines.

Corollary 10. If a pair of opposite chords of intersection of each of two conics with a third meet at the same point, that is, if the four lines are concurrent, a pair of opposite chords of intersection of the two conics with each other pass through the point.

Grouping each of the two conics with its given chords of intersection with the third conic, we have four conics obeying the hypothesis of the main theorem. Hence there is an eight-point conic through the intersections of the two conics with each other and the intersections of the chords with each other. But since the four points of intersection of the chords with each other are coincident the eight-point conic either intersects itself or is tangent to itself. Hence it is degenerate and the corollary is established.

It is apparent that corollary 8 is a special case of the above.

Corollary 11. If in corollary 2 the two straight lines are coincident we have the following: If different segments of the same straight line are chords of contact of two pairs of conics having double contact, then the sixteen points of intersection of the two pairs of conics with each other are in two groups of eight each on two additional conics.

Several special cases of this corollary follow.

Corollary 12. The four points of intersections of any two hyperbolas with each other are on a conic through the four points of intersection of their asymptotes with each other.

This follows directly from corollary 11 and from the fact that any hyperbola and its asymptotes have double contact, the chord of contact being the line at infinity. The other eight-point conic in this case gives the following:

Corollary 13. The four points of intersection of one hyperbola with the asymptotes of a second are on a conic through the four points of intersection of the second hyperbola with the asymptotes of the first.

Corollary 14. Given two pairs of concentric circles; the eight points of intersection of the circles with each other are in two groups of four each on two additional circles.

This follows from the fact that concentric circles have double contact on the line at infinity. Moreover the eight missing points of intersection are the circular points at infinity and thus the conics of the conclusion are circles.

Corollary 15. It is obvious that a hyperbola with its asymptotes and two concentric circles may also be used to discover two eight-point conics as in corollaries 12 to 14.

Corollary 16. Given a tangent to a parabola and two conics having double contact, the chord of contact being a segment of that line parallel to the axis of the parabola which passes through the point of contact of the tangent and parabola; the intersections of the parabola with either conic and of the tangent with the other are on a conic. The species of this conic is the same as the species of the given conic which meets it on the tangent. If this conic is a circle so is the resulting conic.

This corollary is seen to be true from the following considerations: The para-

bola has double contact with the tangent and the line at infinity, the parallel to the axis being the chord of contact. Hence corollary 11 may be applied to this case. The nature of the points of intersection of the given conics with the line at infinity fixes the species of the resulting conics since these points are on the resulting conics.

BOOK REVIEWS.

UNDER THE DIRECTION OF W. H. BUSSEY.

The Teaching of Arithmetic. By DAVID EUGENE SMITH. Ginn and Company, Boston, 1913. v+196 pages.

A Textbook on the Teaching of Arithmetic. By ALVA WALKER STAMPER. American Book Company, New York, 1913. 284 pages.

Considerable attention should be paid to the teaching of arithmetic by all of those who have in charge the training of teachers and administrative officers of the public school system. Hence the universities as well as the normal and training schools should provide instruction along these lines. Both of these texts under discussion are well adapted for this purpose and it is to be hoped that they will enjoy wide use.

Eleven years ago Professor Smith and Professor McMurry of Teachers College prepared for the Teachers College Record an able article on the teaching of arithmetic. This was well adapted for instruction purposes and was so used until the edition was exhausted. In 1909 Professor Smith prepared for the same journal a new article with the same title and used some of the same material. This was also published in book form by Teachers College. The latter has been somewhat revised and expanded for this work from the press of Ginn and Co. The discussion of number games, briefly treated before, has been elaborated into a useful chapter. The chapter entitled "Subjects for Experiment" deserves the close attention of educators. Similar suggestions have been made in the MONTHLY for such experiments in connection with the high school and college mathematics. In this treatment some reference should certainly have been made to the tests for measurement of the efficiency of instruction in arithmetic, notably those suggested by S. A. Courtis. As a part of the recent examination of the New York City Schools such tests were used and in schools all over the country similar tests are being made. A discussion by Professor Smith of the value of these tests would have been a valuable and timely addition to the work.

The second of these works, by A. W. Stamper, takes up the topics much more in detail and does not treat the larger and more general phases of the subject. For this reason this book may be found more useful by normal schools and training classes. Some of the historical matter in this text is misleading. Thus it is quite incorrect to say that "the science of algebra originated in the fourth century B. C. in the Greek city of Alexandria in Egypt." Sciences do not originate in this definite and precise way and Alexandria is not connected with its origin

as intimately as Egypt and possibly India. Similarly trigonometry did not have its initial development in that city. A serious fault in this work seems to the reviewer to be the neglect to emphasize the decimal character of our number system and the import of this fact for the instruction in arithmetic. Constant emphasis on the decimal character of our number system, and use of it, greatly simplifies the work of computation and leads to a better understanding of the processes of reckoning. For instance, on page 50 it is suggested that in mentally adding, 85, 25, and 40, we should first add 15 to 85 making 100, then add the remaining 10 of 25, and finally add 40. This, it seems, is contrary to all well known methods of rapid mental computation which universally proceed from left to right. In this problem the child should be taught to first add 20 to 80, then add 40, and finally add the two remaining fives. These suggestions may seem to be trifles but it is precisely in these trifles that pupils obtain or lose comprehension of the processes of arithmetic.

Both authors recommend the so-called Austrian, or additive, method of subtraction but neither one presents the application of this system to problems in long division in which, instead of writing down the partial products, one writes only the successive *remainders*. Thus to divide 13650 by 243 the plan is as follows:

$$\begin{array}{l} 243)13650(56 \\ \qquad\quad 150 \\ \qquad\qquad 42 \end{array}$$

$5 \times 3 = 15$; $5 + \mathbf{0} = 5$. $5 \times 4 = 20$; $20 + 1$ (carried) $= 21$; $1 + \mathbf{5} = 6$. $5 \times 2 = 10$; $10 + 2$ (carried) $= 12$; $12 + \mathbf{1} = 13$. The 0, 5, 1 are set down in turn and form the remainder when 5×243 is subtracted from 1365. Proceeding in the same way the next product 6×243 is not set down but simply the remainder. In most European schools where the additive method is taught this form of long division is suggested. It is easily mastered and saves much of the *longness* of long division.

Both of these works are notable and desirable additions to the literature on instruction in arithmetic. Mathematicians have somewhat neglected this field and, partly in consequence, the erroneous notion has become rather widespread that the less an individual knows about mathematics the better qualified he is to write on arithmetic and instruction in arithmetic. These works present the material from the standpoint of the trained mathematician who is at the same time familiar with the elementary schools. Their influence will be felt in elementary instruction.

L. C. KARPINSKI.

Analytic Geometry and Principles of Algebra. By ALEXANDER ZIWET, Professor of Mathematics at the University of Michigan, and LOUIS ALLEN HOPKINS, Instructor in Mathematics at the University of Michigan. New York, The Macmillan Company, 1913. In the series edited by E. R. Hedrick. viii+369 pages and 150 figures.

As far back as the oldest inhabitant can remember it has been considered the proper thing to write text-books, and publishers are usually eager to assist the

unselfish missionary in presenting his views as to the proper manner of educating the masses. All too frequently the prospective author has responded to this high call before being kissed by the muse, and has been obliged to profit by the efforts of his predecessors to such an extent that each successive contribution secures an ever increasing degree of uniformity.

According to the historical researches of Heine, the massive wall was built about the city of Göttingen to prevent some enterprising docent from bringing new ideas into the university. To a large extent, so far as text-book writing is concerned, a very efficient wall of conservatism had been erected.

And then came a reaction; side by side with this conservative element a new faction was developed which represented the other extreme. We have algebras for boys, algebras for girls; geometry for engineers, geometry for medical students, and so on; probably we may soon expect trigonometry for red-headed students and even further differentiation.

Finally, once in a while we see the announcement of a text-book written by really chosen authors—persons of broad scholarship and ripe experience who undertake the task with a clearly defined purpose, having no axe to grind and fearless of charges of being too old or too new.

Since the book under review belongs distinctly to the third category, it demands more than a passing notice. The mechanical make-up of the book merits favorable comment; the large type, spaced lines, consistent use of italics, bold-face etc., the well executed figures, the full index, and the complete answers not only make for easy reading but put a big premium on neatness and orderliness on the part of the reader.

As the title indicates, the volume is concerned not only with analytic geometry, but also with the principles of algebra. Elementary algebra, plane and solid geometry and some knowledge of trigonometry are presupposed, but nothing of graphical processes nor of the subjects of advanced algebra.

The text begins much as the ordinary text on analytics, but the persistent use of $\Delta x = x_2 - x_1$ and the early introduction of determinants are innovations. Projection, polar coördinates, and vectors are all discussed before the equation of the straight line is mentioned. The projective forms of the equation are treated in great detail, that of one line being followed by that of a pencil, including the point and angle of intersection of two given lines. Here follows a second, more extensive development of determinants, including co-factors, minors, and elimination. Finally the normal form of the equation of a line is derived from the polar equation, and the usual applications made. Then follows a whole chapter on permutations and combinations, mathematical induction, and a systematic treatment of determinants, including multiplication.

In the chapter on the circle the ordinary treatment is enlivened by the use of determinants, the discussion of quadratic equations, inversion, pole and polar, pencils of circles and parametric representation. Since the quadratic equation may involve imaginaries, a full discussion of numbers, rational, irrational, fractional, complex, together with a geometric interpretation of each case is

given. This chapter closes with a proof of DeMoivre's theorem. A well written chapter on the properties and graphical representation of higher polynomials includes the concepts of the first and second derivatives, maxima and minima, points of inflexion, roots of numerical equations, both rational and irrational, Horner's method, and the expansion of a polynomial by Taylor's series. It comprises everything usually provided in the theory of equations, yet comes in so naturally that it seems a necessary part of analytic geometry.

The chapter on the parabola adds to the topics of the traditional course the parametric representation, area of a segment, and a full derivation of Simpson's rule for the approximate area under any curve, also an application to shearing force and bending moment.

The two chapters on central conics are more conservative; they are first discussed from the definition of the locus of a point the sum (or difference) of whose distances from two fixed points is a constant, and later by means of the eccentricity, in which it is shown that all conics can be cut from a right circular cone.

Then follows a chapter on algebraic curves, including the determination of the number of constants, the equations of tangents and normals, double points at the origin, and a detailed study of a number of particular algebraic and transcendental curves, incidentally developing the theory of logarithms. The principles of this chapter are applied to the study of empirical equations and a number of practical suggestions are added regarding cross-section paper, logarithmic paper, etc.

The part on solid geometry is very brief. The positive directions of the coördinate axes are so chosen that the xy-plane as viewed from the positive z-axis is the same as the xy-plane as previously studied. Linear systems of planes are treated fully, including the derivation of the volume of the tetrahedron determined by four non-coplanar points. In the discussion of the sphere we find inversion, tangent cone, polar theory and linear systems touched upon; the other quadric surfaces are but briefly treated, yet ruled surfaces are considered, and contour lines mentioned. Rotation of axes is all put in small type; it includes the derivation of Euler's formulas. A short appendix illustrates abridged multiplication and division.

Within the compass of this volume are found most of the subjects understood under the heading of a first course in analytic geometry, an almost complete treatment of advanced algebra, and a wealth of other material not found in either. And the book nowhere seems to be overloaded, but every page is lively and attractive.

The changes from one subject to another are not digressions, but are made consistently with the sound pedagogic principle that each idea is developed as needed, whether the idea comes from the box labeled algebra, or calculus, or what not.

A reviewer is always supposed to find some fault, just to show how much wiser he is than the authors of the book under review, but in this case the reviewer feels a certain embarrassment, and even detects symptoms of a wish to have

been the author himself. However, a few points may be mentioned. In discussing the bisectors of the angles between lines or between planes, the only method derived is valueless if one of the boundaries of the angle passes through the origin. Incidentally, does any book treat this subject properly? The whole discussion of positive and negative regions, so well done for curves, is hardly convincing when applied to lines and to planes. A considerable number of the later proofs are not complete, and in some cases rather too brief to be of great service, yet with proper skill on the part of the instructor may have influence in the right direction.

In those institutions in which advanced algebra is required for entrance, the book would either have to be abbreviated, or a number of topics repeated. Perhaps it is an argument to allow more alternatives for entrance, and make frank provision for teaching algebra to all in college.

VIRGIL SNYDER.

PROBLEMS AND SOLUTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

Special Notice.—Please reread the requests as to form of solutions on pp. 258–259 of the October 1913 issue. Unless these directions are observed by contributors, solutions must either be entirely rewritten by the committee or else rejected. Put all drawings on separate sheets.

MANAGING EDITOR.

ALGEBRA.

When this issue was made up no solutions had been received for numbers 401, 402, 404, and 406. Please give attention to these.

405. Proposed by E. J. MOULTON, Northwestern University.

Given the alternating series

$$s = 1 - 1/2 + 1/3 - 1/4 + 1/5 - \cdots.$$

(*a*) Let s_n be the sum of the first n terms of the series. Show that in order to make the difference $s - s_n$ numerically less than $1/2k$ (k a positive integer) it is necessary and sufficient to take $n = k$; hence s_{500} differs from s by less than .001. (*b*) Let s_n' be the sum of s_n and 1/2 the $(n + 1)$th term of the series. Show that the difference $s - s_n'$ is numerically less than $1/2n(n + 1)$; hence s_{22}' differs from s by less than .001.

406. Proposed by S. A. COREY, Hiteman, Iowa.

Solve the system of equations:

$$\begin{aligned}(1 - x)(a_1 + a_2y + a_3z) &= d,\\ (1 - y)(b_1 + b_2x + b_3z) &= g,\\ (1 - z)(c_1 + c_2x + c_{3y}) &= h.\end{aligned}$$

407. Proposed by E. B. ESCOTT, University of Michigan.

In computing the values of the natural logarithms of 2, 3, and 5 by the following formulas:

$$\begin{aligned}\log 2 &= 2(7P + 5Q + 3R),\\ \log 3 &= 2(11P + 8Q + 5R),\\ \log 5 &= 2(16P + 12Q + 7R),\end{aligned}$$

where P, Q, and R are numbers which were computed by infinite series (G. Chrystal, *Algebra*, Part II, chapt. 28), it is found, on comparing the results with the known values of these logarithms to 15 decimals, that there are the following errors: $-2{,}533$, $-4{,}052$, and $6{,}080$, respectively. Find the errors in P, Q and R.

GEOMETRY.

When this issue was made up no solutions had been received for numbers 410, 417, 421, and 427. Please give attention to these.

434. Proposed by CLIFFORD N. MILLS, Bloomington, Ind.

ABC is any triangle with sides a, b, c. Prove by purely geometrical methods that the area $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = 1/2(a+b+c)$. (From Olney's *Geometry*, page 255, with slightly simplified notation.)

435. Proposed by R. M. MATHEWS, Riverside, California.

From a fixed point A perpendiculars are dropped to the tangents drawn to a circle whose center is O. Prove that the locus of the feet of the perpendiculars is a limaçon.

436. Proposed by A. J. KEMPNER, University of Illinois.

Given in a plane two similar curves arbitrarily situated, except that they shall possess the same sense of direction (which, of course, does not mean that they are similarly located). Let corresponding points on both curves be joined by straight lines, and let all of these straight lines be divided in the same ratio $\lambda : 1$, λ being any real number. Prove that the points of division all lie on a curve similar to the two given curves except when they all happen to fall together.

CALCULUS.

When this issue was made up no solutions had been received for numbers 335, 337 to 340, 342, 348, and 350. Please give attention to these.

356. Proposed by B. F. FINKEL, Drury College.

A steel girder l feet long and w feet wide is moved on rollers along a passageway a feet wide and into a corridor at right angles to the passageway. How wide must the corridor be to just admit the girder?

357. Proposed by W. D. CAIRNS, Oberlin College.

A continuous variable represented by a point on a vertical line changes according to such a law that it is reduced to $1/m$ of its value on being moved a units upward, irrespective of the special position from which it is moved. Find the law of change, that is, the relation between the variable y and the height h of the variable point above a fixed point of the vertical line.

358. Proposed by C. N. SCHMALL, New York City.

About a given circle circumscribe the smallest parabola.

MECHANICS.

When this issue was made up no solutions had been received for numbers 271 to 275 inclusive, and 277 to 279 inclusive. Also 268 and 269, proposed in 1912, have not been solved. Please give attention to these.

288. Proposed by C. E. HORNE, Westminster College, Colorado.

Show that the tangential velocity of a projectile at any point of its path is equal to the velocity it would have acquired in falling, under the influence of gravitation alone, from the directrix to the point in question (Hulburt's *Calculus*, p. 220, ex. 10).

289. Proposed by C. N. SCHMALL, New York City.

A particle of elasticity e is projected with a velocity v at an angle of elevation ϕ from a point on a smooth horizontal plane. Show that after $2v \sin \phi/g(1-e)$ seconds, it will cease to rebound and will move along the plane with a uniform velocity $v \cos \phi$.

NUMBER THEORY.

When this issue was made up no solutions had been received for numbers 189, 191, 192, 196, 201, and 202. Please give attention to these.

208. Proposed by E. T. BELL, Seattle, Washington.

If an odd number is perfect, it cannot be the sum of two squares.

209. Proposed by R. D. CARMICHAEL, Indiana University.

Prove that the difference of the sixth powers of two integers cannot be the square of an integer

210. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

If a and b are relatively prime and $(a + b)$ is even, then $(a - b)ab(a + b) \equiv 0$ (mod. 24).

SOLUTIONS OF PROBLEMS.

ALGEBRA.

394. Proposed by E. B. ESCOTT, Ann Arbor, Michigan.

Solve the equation $\sin^2 x \sin^2 2x = 5/16$.

SOLUTION BY R. M. MATHEWS, Riverside, California.

Taking the square root of each side of the given equation we obtain

$$(1)\ \sin x \sin 2x = \frac{\sqrt{5}}{4} \quad \text{and} \quad (2)\ \sin x \sin 2x = -\frac{\sqrt{5}}{4}.$$

From (1),

$$2 \sin^2 x \cos x = \frac{\sqrt{5}}{4}, \quad \text{or} \quad (3)\ \cos^3 x - \cos x + \frac{\sqrt{5}}{8} = 0.$$

Inspection shows that $-\dfrac{\sqrt{5}}{2}$ is a value of $\cos x$ in (3). Eliminating this impossible solution, we have the quadratic

$$(4) \qquad \cos^2 x - \frac{\sqrt{5}}{2} \cos x + \frac{1}{4} = 0,$$

the roots of which are $\cos x = \dfrac{\sqrt{5} + 1}{4}$ and $\cos x = \dfrac{\sqrt{5} - 1}{4}$.

The solutions of (2) are the negatives of those of (1) just found.

The side of a regular inscribed decagon is $(\sqrt{5} - 1)/2$, whence $(\sqrt{5} - 1)/4 = \cos 72°$. From this, it is easily shown that $(\sqrt{5} + 1)/4 = \cos 36°$.

Since in the original equations the functions are all squared, the signs which they may have in particular quadrants are immaterial. Hence

$$x = n\pi \pm \tfrac{1}{5}\pi, \quad x = n\pi \pm \tfrac{2}{5}\pi,$$

where n is any positive or negative integer, are the real values satisfying the equation.

Also solved by C. E. GITHENS, HORACE OLSON, ALBERT R. NAUER, RICHARD MORRIS, A. M. HARDING, ELMER SCHUYLER, S. W. REAVES, W. C. EELLS, LEROY M. COFFIN, A. H. HOLMES, F. M. MORGAN, H. C. FEEMSTER, J. L. RILEY, H. E. TREFETHEN, and the PROPOSER.

395. Proposed by V. M. SPUNAR, Chicago, Illinois.

Solve the system of equations

$$x_1^2x_2 = a_1, \quad x_2^2x_3 = a_2, \quad x_3^2x_4 = a_3, \quad \cdots, \quad x_n^2x_1 = a_n.$$

I. SOLUTION BY THOMAS E. MASON, Indiana University.

Assuming each a different from zero and expressing each x in terms of x_1 from the equations in order, we have

$$x_2 = \frac{a_1}{x_1^2},\ x_3 = \frac{a_2 x_1^4}{a_1^2},\ x_4 = \frac{a_3 a_1^4}{a_2^2 x_1^8},\ \cdots,\ x_i = \frac{a_{i-1} a_{i-3}^4 a_{i-5}^{16} \cdots}{a_{i-2}^2 a_{i-4}^8 a_{i-6}^{32} \cdots} x_1^{(-2)^{i-1}}.$$

Continuing this substitution we would obtain

$$x_{n+1} = \frac{a_n a_{n-2}^4 a_{n-4}^{16} \cdots}{a_{n-1}^2 a_{n-3}^8 a_{n-5}^{32} \cdots} x_1^{(-2)^n}.$$

But from the last of the given equations, we see that x_{n+1} is defined to be x_1. Hence we have

$$x_1 = \frac{a_n a_{n-2}^4 a_{n-4}^{16} \cdots}{a_{n-1}^2 a_{n-3}^8 a_{n-5}^{32} \cdots} x_1^{(-2)^n}, \quad \text{or} \quad x_1^{2^n+(-1)^{n-1}} = \left(\frac{a_n a_{n-2}^4 a_{n-4}^{16} \cdots}{a_{n-1}^2 a_{n-3}^8 a_{n-4}^{32} \cdots} \right)^{(-1)^{n-1}}.$$

This can be solved by the ordinary trigonometric method and the $2^n + (-1)^{n-1}$ roots found. For each value of x_1 there will be corresponding values of x_i found by substituting the value of x_1 in the formula above for x_i.

Solved in like manner by H. C. FEEMSTER, A. H. HOLMES, and S. W. REAVES.

II. SOLUTION BY F. C. REISLER and I. A. BARNETT, University of Chicago.

Given the system of equations:

$$x_1^2 x_2 = a_1,\ x_2^2 x_3 = a^2,\ \cdots,\ x_n^2 x_1 = a_n.$$

If no $a = 0$, the following law is found to be true for all cases through $n = 4$.

$$x_k^{2^n-(-1)^n} = \prod_{l=0}^{n-1} a_{k+l}^{2^{n-1-l}(-1)^l}, \tag{1}$$

where $k = 1 \cdots n$, and a_{n+q} is defined as a_q.

In order to show that (1) holds true for all values of n, we assume that it holds for $n = m$, that is,

$$x_k^{2^m-(-1)^m} = \prod_{l=0}^{m-1} a_{k+l}^{2^{m-1-l}(-1)^l}, \tag{2}$$

and prove that it holds for $n = m + 1$.

To do this we take a new set of $m + 1$ equations

$$y_1^2 y_2 = a_1,\ y_2^2 y_3 = a_2,\ \cdots,\ y_m^2 y_{m+1} = a_m,\ y_{m+1}^2 y_1 = a_{m+1}.$$

From these and the given equations we obtain the new system,

$$x_1^2 x_2 = y_1^2 y_2,\ x_2^2 x_3 = y_2^2 y_3,\ \cdots,\ x_m^2 x_1 = y_m^2 y_{m+1},\ a_{m+1} = y_{m+1}^2 y_1, \tag{3}$$

where y_{m+1+p} is defined as y_p.

The solution of this system of equations is given by (2) as follows:

$$x_k^{2^m-(-1)^m} = \prod_{l=0}^{m-1} [y_{k+l}^2 y_{k+l+1}]^{2^{m-l-1}(-1)^l}.$$

In this product all of the factors except the first in the first couplet and the last in the last couplet become unity when combined in pairs, as can be seen by examining any such pair, for example,

$$(y_{k+l+1})^{2^{m-l-1}(-1)^l}(y_{k+l+1}{}^2)^{2^{m-l-2}(-1)^{l+1}},$$

in which the second factor may be written $(y_{k+l+1})^{2^{m-l-1}(-1)^{l+1}}$, and hence the sum of the exponents is zero no matter whether m is odd or even.

Therefore

$$x_k{}^{2^m(-1)^m} = y_k{}^{2^m} y_{k+m}{}^{(-1)^{m-1}} = \prod_{l=0}^{m-1} a_{k+l}{}^{(-1)^l 2^{m-1-l}}. \tag{4}$$

Solve (4) for $y_k{}^{2^m}$, square both sides, put $y_{m+k}{}^2 = (a_{m+k})/y_k$ as given by the last of equations (3), and we obtain, after a little reduction,

$$y_k{}^{2^{m+1}+(-1)^m} = \prod_{l=0}^{m} a_{k+l}{}^{(-1)^l 2^{m-l}}. \tag{5}$$

We see that (5) is the same as (2) with m replaced by $m+1$. Hence, if (1) holds for $n = m$, it also holds for $n = m+1$. But it has been verified that (1) holds for $n = 1, 2, 3, 4$. Hence it holds for all positive integral values of n.

In addition to the two interesting solutions of this problem given above, still another solution is worthy of reproducing, though the author is unknown, having neglected to *attach his name to his work.* If he sent a letter, it became separated from the solution. Will contributors *please follow the printed form* in sending in solutions. Also see full suggestions in October, 1913, issue.

III. Taking the logarithm of each member of each equation, we have the following set of n simultaneous equations in n unknowns.

$$\begin{array}{l}
2\log x_1 + \log x_2 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots = \log a_1 \\
\cdots\cdots\cdots 2\log x_2 + \log x_3 \cdots\cdots\cdots\cdots\cdots = \log a_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots 2\log x_3 + \log x_4 \cdots\cdots = \log a_3 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\log x_1 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + 2\log x_n = \log a_n.
\end{array}$$

Then, by principles of determinants, we have

$$\log x_1 \begin{vmatrix} 2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 1 & 0 & \cdots & 0 \\ \cdots & & & & & \cdots \\ \cdots & & & & & \cdots \\ 0 & 0 & \cdots & \cdots & 2 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 2 \end{vmatrix} = \begin{vmatrix} \log a_1 & 1 & 0 & \cdots & 0 \\ \log a_2 & 2 & 1 & \cdots & 0 \\ \cdots & & & & \cdots \\ \cdots & & & & \cdots \\ \vdots & & & & \\ \log a_n & 0 & \cdots & \cdots & 2 \end{vmatrix}$$

. . . .

. . . .

$$\log x_j \begin{vmatrix} 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 2 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 & \cdots & \log a_1 & 0 & \cdots & 0 \\ 0 & 2 & 1 & \cdots & \log a_2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \log a_j & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & \cdots & \log a_n & 0 & \cdots & 2 \end{vmatrix}.$$

Denoting the determinant in the left member of the last equation by D and the one in the right member by A_j, we have

$$\log x_j = \frac{A_j}{D},$$

or

$$x_j = e^{\frac{A_j}{D}} \; (j = 1, 2, \cdots, n).$$

Also solved by the PROPOSER, who used logarithms as in the last solution, though he did not use determinants.

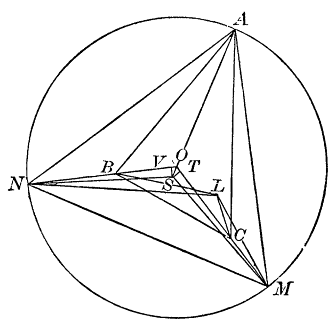
GEOMETRY.

424. Proposed by H. E. TREFETHEN, Colby College.

In a given triangle ABC, determine by geometric demonstration the point O such that the sum of the distances, $AO + BO + CO$, shall be a minimum.

SOLUTION BY M. E. GRABER, Heidelberg University.

On each of the three sides of the triangle ABC, describe a segment of a circle to contain an angle of 120°. The arcs of these circles intersect at a point O, about which the angles are each 120°. Then $AO + BO + CO$ is a minimum. Suppose $OA > OB$ or OC. Using OA as a radius describe a circumference. Produce OC and OB to meet the circumference at M and N. Let L be any other point in ABC.



Draw LC, LM, LB, LN, AM, MN, NA. Then $AM = MN = NA$ and $LA + LN + LM > OA + ON + OM$. In order to prove this relation, it is evidently sufficient to prove that $OA + ON + OM <$ the sum of the distances from S (the vertex of an isosceles Δ on MN with the same altitude as ΔLMN)

to A, M, and N. With MS and NS as radii describe arcs cutting MO and NO in T and V. Join T and S, V and S, by straight lines. Then $\angle\ OVS$ is obtuse and therefore $\angle\ OSV$ is $< 30°$. When $\angle\ SOV = 60°$ and $\angle\ OSV = 30°$, $OV = \frac{1}{2}OS$. When $\angle\ OSV < 30°$, $OV < \frac{1}{2}OS$ and $OV + OT < OS$. Hence $SM + SN + SA > AO + OM + ON$ and $OM + ON + OA < LM + NL + LA$. Since $LC + CM > LM$ and $LB + BN > LN$, we have $LC + CM + LB + BN + LA > OA + OB + BN + OC + CM$. Hence, $LA + LB + LC > OA + OB + OC$.

Also solved by C. N. Schmall, David L. MacKay, A. M. Harding and Harvy Roeser.

CALCULUS.

336. Proposed by EVA S. MAGLOTT, Ada, Ohio.

If a right cone stands on an ellipse, prove that its superficial area is $\frac{\pi}{2}(OA + OA')(OA \cdot OA)'^{\frac{1}{2}}$ times $\sin\alpha$, where O is the vertex of the cone, A and A' extremities of the major axis of the ellipse, and α is the semi-angle of the cone.

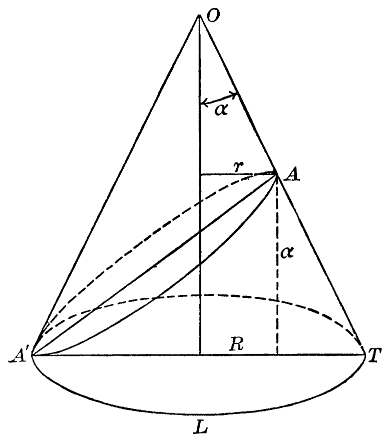
SOLUTION BY ELMER SCHUYLER, Brooklyn, N. Y.

The formula for the surface $AA'LT$ is

$$\frac{\pi}{R-r}\sqrt{a^2+(R-r)^2}\,\{R^2-\tfrac{1}{2}(R+r)\sqrt{Rr}\}.$$

(See Finkel's *Mathematical Solution Book*, pp. 318–319).

Here $r = OA\sin\alpha$, $R = OA'\sin\alpha$, and $a = (OA' - OA)\cos\alpha$.



For brevity, put $OA = K$, $OA' = K'$. Then this formula becomes, after slight reduction,

$$\frac{\pi}{\sin\alpha}\cdot\sin^2\alpha\{K'^2-\tfrac{1}{2}(K+K')\sqrt{KK'}\} = \frac{\pi}{2}\sin\alpha\,\{2K'^2-(K+K')\sqrt{KK'}\}.$$

The surface $OA'A$ is the total conical surface less the ungula, that is,

$$\frac{\pi}{2}\sin\alpha[2K'^2 - 2K'^2 + (K+K')\sqrt{KK'}] = \frac{\pi}{2}(K+K')(KK')^{\frac{1}{2}}\sin\alpha.$$

The problem rests upon the tedious integration of the form,

$$\int_r^R 2x\cos^{-1}\left[\frac{2Rr-(R+r)x}{(R-r)x}\right]dx.$$

Also solved by A. M. HARDING and P. PENALVER.

A solution of 334 was received from ELMER SCHUYLER, after the the forms were sent to the printer.

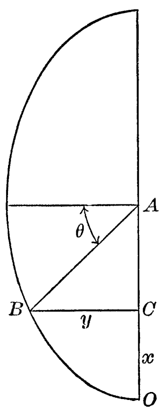
MECHANICS.

270. Proposed by W. J. GREENSTREET, Burghfield, England.

A cycloid has its base vertical. Find the line of quickest descent from the middle point of the base, and its approximate inclination to the horizon.

SOLUTION BY J. SCHEFFER, Hagerstown, Md.

Let AB be any straight line drawn from the middle point A of base of the cycloid, and let θ be the angle that it makes with the horizon. The time of descent from A to B is $t=\sqrt{\frac{2}{g}\cdot\frac{s}{\sin\theta}}$, AB being $=s$. But $s^2=(r\pi-x)^2+y^2$, x and y being subject to the equations of the cycloid, $x=r(\phi-\sin\phi)$, $y=r(1-\cos\phi)$. Substituting, we have $s^2=r^2[(\pi-\phi)+\sin\phi]^2+(1-\cos\phi)^2$. We have to get the minimum of $s/\sin\theta$, which reduces to



$$\frac{(\psi+\sin\psi)^2+(1+\cos\psi)^2}{\psi+\sin\psi},$$

after putting $\pi-\phi=\psi$ and omitting the constant r^2.

Differentiating and setting the differential coefficient equal to zero, we get, after some easy reductions, the transcendental equation $\psi^2-2\cos\psi-2=0$. Whence $\psi=\pm 2\cos\psi/2$, and using the positive sign, we find $\psi=85°$, nearly. Then

$$\tan\theta=\frac{r\pi-x}{y}=\frac{\psi+\sin\psi}{1+\cos\psi}.$$

Hence, finally, $\theta = 53° 39'$, nearly.

Note. It is assumed in the above solution that the line sought is the *straight line* of quickest descent. Otherwise the problem becomes much more difficult, requiring for its solution the calculus of variations. THE EDITORS.

NUMBER THEORY.

A solution of 188 was received from ELMER SCHUYLER after the December issue had been made up.

194. Proposed by L. E. DICKSON, University of Chicago.

Find two numbers a and b each of two digits such that, when the product ab is found by the usual method, the two partial products to be added are exactly the same as the partial products in getting the product ba. Is there a similar pair of numbers of three digits?

SOLUTION BY B. F. YANNEY, University of Wooster.

Let $a = 10x_2 + x_1$, $b = 10y_2 + y_1$.

Then equating corresponding partial products of ab and ba, we get, after reducing, $x_1/y_1 = x_2/y_2$ in each case. Hence this is a necessary condition. Obviously, it is also a sufficient condition.

Examples: $a = 21,\ 32,\ 43;$
$b = 84,\ 96,\ 86.$

In the case of numbers of three digits, let

$$a = 10^2x_3 + 10x_2 + x_1, \quad b = 10^2y_3 + 10y_2 + y_1.$$

Again equating corresponding partial products and reducing, we get

$$10\,x_3y_1 + x_2y_1 = 10\,x_1y_3 + x_1y_2, \tag{1}$$

$$10^2x_3y_2 + x_1y_2 = 10^2x_2y_3 + x_2y_1, \tag{2}$$

$$10\,x_2y_3 + x_1y_3 = 10x_3y_2 + x_3y_1, \tag{3}$$

Since all the digits are less than 10, it is obvious in (2) that $x_3y_2 = x_2y_3$ and $x_1y_2 = x_2y_1$. Whence we get

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3}. \tag{A}$$

Furthermore, (A) substituted in (1) and (3) gives identities as results. Hence (A) is a necessary condition for the numbers a and b to be such as are required. Obviously also (A) is a sufficient condition. Example: $a = 214$, $b = 428$.

In conclusion we state a general theorem, suggested by the above considerations, the proof of which is not difficult.

THEOREM. Two numbers a and b of n digits each are such that, when the products ab and ba are found by the usual method, the corresponding partial products to be added are exactly the same, if and only if the corresponding digits of the two numbers are proportional.

Illustration:

$$\begin{array}{rcr}
1342 & & 2684 \\
\underline{2684} & & \underline{1342} \\
5368 & = & 5368 \\
10736 & = & 10736 \\
8052 & = & 8052 \\
2684 & = & 2684
\end{array}$$

Solved in similar manner by H. C. FEEMSTER, S. W. REAVES, W. C. EELLS, C. E. GITHENS, A. H. HOLMES, and V. M. SPUNAR.

195. Proposed by E. B. ESCOTT, University of Michigan.

Find triangles whose sides are integers and one of whose angles is 60°.

SOLUTION BY ARTEMAS MARTIN, Washington, D. C.

By trigonometry we have for *any* triangle the equation

$$x^2 - 2xy\cos\phi + y^2 = z^2, \tag{1}$$

in which x and y denote the sides including the angle ϕ, and z the side opposite that angle.

In this equation ϕ is supposed to be a given angle and x, y, z wholly unknown and are to be determined in whole numbers.

When $\phi = 60°$, (1) reduces to

$$x^2 - xy + y^2 = z^2. \tag{2}$$

Assuming $z = x - \dfrac{qy}{p}$ we get from (2), after some reduction,

$$\frac{y}{x} = \frac{p^2 - 2pq}{p^2 - q^2};$$

therefore we may take

$$x = p^2 - q^2 \text{ and } y = p^2 - 2pq.$$

Substituting these values of x and y in the assumed value of z we find

$$z = p^2 - pq + q^2.$$

Solving (2) for y we get

$$y = \tfrac{1}{2}x \pm \sqrt{z^2 - \tfrac{3}{4}x^2}. \tag{3}$$

This result shows that y has two values, and that there are two triangles for each value of x and z.

Substutiting the values of x and y found above in (3) we get

$$\begin{aligned} y &= \tfrac{1}{2}(p^2 - q^2) \pm \tfrac{1}{2}(p^2 - 4pq + q^2), \\ &= p^2 - 2pq \text{ or } 2pq - q^2. \end{aligned}$$

It can be shown by a geometrical construction that y has two values.

From what has been done it is obvious that the required triangles for each value of p and q are

$$p^2 - q^2, \quad p^2 - 2pq, \quad p^2 - pq + q^2;$$

and

$$p^2 - q^2, \quad 2pq - q^2, \quad p^2 - pq + q^2;$$

where p can have any value, but q is limited by the following restrictions:

1. q must be less than p and prime to it;
2. q must not be greater than $\frac{1}{2}(p - 1)$ when p is odd, and must be less than $\frac{1}{2}p$ when p is even;
3. $p + q$ must not be divisible by 3, as in that case the sides of the triangle will be divisible by 3.

The following table contains the first seven pairs of such triangles.

p	q	Sides		
		$p^2 - q^2$	$p^2 - 2pq$, $2pq - q^2$	$p^2 - pq + q^2$
3	1	8	3	7
3	1	8	5	7
4	1	15	8	13
4	1	15	7	13
5	2	21	5	19
5	2	21	16	19
6	1	35	24	31
6	1	35	11	31
7	1	48	35	43
7	1	48	13	43
7	3	40	7	37
7	3	40	33	37
8	3	55	16	49
8	3	55	39	49

Also solved by THOMAS E. MASON, A. H. HOLMES, B. F. YANNEY, H. E. TREFETHEN, HORACE OLSON, N. A. DRAXTEN, E. B. ESCOTT, and C. E. GITHENS.

MISCELLANEOUS QUESTIONS.

EDITED BY R. D. CARMICHAEL.

In the teaching of mathematics there are three fields of experience, now more or less separated, which it is desirable to have joined into one; namely, that of teachers in the better high schools, that of teachers in the smaller colleges and that of teachers in the larger and more powerful institutions. The problems in these fields are not as widely separated or as distinct as some have supposed. We need to find a means of funding the experience from all these three sources so that it may become the common possession of all who are at work in these fields.

It is our desire that this department of the MONTHLY shall become an effective means to this end. For this we need the coöperation of a large number of persons engaged in the teaching of mathematics. In this connection there are at least three ways in which our readers may render valuable service to the cause of mathematical education:

(*a*) Give us suggestions as to the plans along which this department should be developed.

(*b*) Propose specific questions on which it is desirable to know what is the general opinion or what is the experience of individual teachers.

(*c*) Respond promptly to the questions printed with a statement of your opinion or an account of your experience or both.

Already we have had a gratifying expression of interest in these matters. We have in hand now an excellent answer to question 5 on culture mathematics and an excellent answer to question 6 on vocational mathematics. Let us have further replies to these and to other questions; let us also have other useful questions proposed. If the interest invoked and the amount of good matter received requires it, we can enlarge this department in later issues so as to meet the demand made upon it.

QUESTIONS.

7. What place should be given to the history of mathematics in courses for prospective high school teachers, and why?

8. One of our correspondents would like to know the experiences of other teachers in giving practice teaching in mathematics during a college course. May we have here the experiences of several teachers?

9. What is the present state of experience with coördinated courses in high school mathematics? What contribution does this promise to the development of mathematics teaching in high schools? What about the corresponding matters in college mathematics? (Note.—An individual correspondent need not answer all the questions in number 9; it is sufficient if he answers only one.)

REPLIES.

Instead of here giving replies to questions on which no communication has yet been printed, we return this time to question 2, giving our space to the following interesting note which was called forth in connection with that question. In this connection it will be of interest to many of our readers to know that the text-book, "The Principles of Projective Geometry," by J. L. S. Hatton, to which reference was made by Mr. Stromquist in the January issue, will be reviewed by h m in the MONTHLY in the near future.

A NOTE ON SYNTHETIC PROJECTIVE GEOMETRY.

By LAO G. SIMONS, Normal College of the City of New York.

The writer of this note is in hearty sympathy with the article by Professor Bussey in the AMERICAN MATHEMATICAL MONTHLY for November, 1913, on "Synthetic Projective Geometry as an Undergraduate Study," and ventures the hope that the experience of a college teacher who has for five years conducted courses simultaneously in synthetic projective geometry and methods of teaching secondary mathematics may be of interest to other teachers of mathematics.

The course in projective geometry in the Normal College is a three-hour course for one semester and follows a set of notes prepared by Dr. George H. Ling, formerly of Columbia University and now at the University of Saskatchewan, based on Doehlemann's Projektive Geometrie in the *Sammlung Göschen.* It includes the following topics in the order named: Part I. Central projection and projection from an axis with the elements at infinity, geometric prime forms, perspective prime forms; principle of duality; the anharmonic ratio; harmonic forms defined metrically and descriptively; complete four-point and four-side with examples of such interest as bisecting a line or an angle by means of a straight edge alone; figures in plane homology; projective relation between prime forms with the geometric construction of the fourth element of an anharmonic ratio, similar and congruent ranges, with such examples as determining a line through the inaccessible intersection of two lines; superposed projective forms with the important theorems on the number of double elements possible and Steiner's geometric determination of these, involution, Desargues's theorem for a complete quadrangle, brought up again later for the circle and conic. Part II. Applications. Geometrical figures (in the plane) generated by projective prime forms,

loci generated by intersections of projective flat pencils, with the problem to find the intersections (if any) of a line with the curve of the second order determined by five points, and the correlative, envelope of a line which moves so as always to join corresponding points of projective ranges, and the problem to find the tangents (if any) through any given point to any curve of the second class which is determined by any five given lines (In passing—Are not this locus and this envelope good subjects for a moving picture?), connection between curves of the second order and curves of the second class; poles and polars, their properties, the self-polar triangle and certain theorems leading to the establishment of the validity of the principle of duality for figures in one plane (concerning which there is some discussion), poles and polars of the elements at infinity. The notes include a chapter on geometrical figures in the sheaf generated by projective prime forms, but the time has been too limited to touch upon this part of the subject. The text includes many exercises, among them theorems necessary to the sequence of the work, an excellent plan for the development of the minds of prospective teachers.

Our pupils are not allowed to take the projective geometry before the end of the second year in college and most of them elect it even later. They are taking as a parallel course the second semester of analytic geometry or they have finished this work, which includes the solid analytic geometry. Only pupils who are really interested in mathematics are advised to take this course. Indeed, a poor student is so strongly "advised" against it that she decides for herself to elect something else. The work is conducted with almost no so-called class recitations. It is assumed that a girl can keep up to the mark so that the possible answering of a few questions at the beginning of the hour enables the class to go right ahead with new work.

And the girls enjoy the work. Term after term, the experience is the same. From the first lesson right through to the last, there is no abatement of interest. The students uniformly do a high order of work in the course, and the rating is the result of four tests during the semester taken in connection with the judgment of the instructor on the individual girl. The opinions of the students expressed to me are that the course has been an inspiration and repeatedly there has been the wish that it might have extended through the year. They feel that the course has given to them the great enjoyment that a mathematics student experiences when he has arrived at a definite conclusion through steps of pure reasoning, when he has proved a fact true that was not at first apparent. It is encouraging that the purely abstract reasoning with no attempt at possible uses for the subject matter gives such pleasure to the college student. The naïve statement of one student was: "The study of projective geometry came nearer to my idea of a recreational course in mathematics than any of the other courses." Personally, I find no more intense interest in any work that I present and no work brings me such real joy and satisfaction.

My judgment of the course is that it broadens, as few college subjects do, the minds of prospective high school teachers. The method of approach is so

different from that of the analytic geometry that the pupils gain at a bound a sense of the variety of mathematical treatment. After having defined a conic as the locus of a point that moves so that its distance from a fixed point bears a constant ratio to its distance from a fixed straight line, after having translated this property into algebraic language, and after having studied the properties of the conic from this equation, it is an illuminating step to find that a conic is the locus of the intersections of corresponding rays of projective flat pencils, that three sets of lines, properly related to each other, may be used to construct these curves that play so important a part in the physical world, and that these curves may be studied from this point of view.

Just one illustration will serve to show what points of contact there are with elementary geometry. In triangles in homology, when the center of homology is at an infinite distance in the direction perpendicular to the axis of homology, and again when the axis of homology is at an infinite distance, the constant in both cases being -1, the two figures are congruent, in reverse order in the first case, and in the same order in the second. These very elements at infinity are a part of the broadening influence of this subject. They may and should be introduced in the analytic geometry but they *must* be employed in the projective geometry; and thus the satisfaction comes in finding theorems that are true under all conditions.

It may be well in closing to recall the report of the Committee of Ten on secondary school studies. "A place should also be found either in the school or college course for at least the elements of the modern synthetic or projective geometry. It is astonishing that this subject should be so generally ignored, for mathematics offers nothing more attractive. It possesses the concreteness of the ancient geometry, without the tedious particularity, and the power of analytical geometry without the reckoning, and by the beauty of its ideas and methods illustrates the esthetic quality which is the charm of the higher mathematics, but which the elementary mathematics in general lacks."

NOTES AND NEWS.

UNDER THE DIRECTION OF FLORIAN CAJORI.

Mr. W. C. EELLS, head of the department of mathematics in Whitworth College, Washington, has been elected instructor in the United States Naval Academy at Annapolis.

The January number of the *Hibbert Journal* contains a short article by Professor C. W. COBB, of Amherst College, on "Certainty in Mathematics and in Theology."

Mr. HYLAND CLAIR KIRK has an article on "The fourth dimension" in the December number of the *Open Court*. It is a humorous satire.

BENJAMIN O. PEIRCE, Hollis professor of mathematics and natural philosophy in Harvard college, died at Cambridge, Mass., January 15, 1914. He was a

member of learned societies and the author of several books on mathematics and physics.

Mr. A. L. MILLER, A.B. 1911, A.M. 1913, of Harvard, for two years instructor in mathematics at Harvard, is now instructor in mathematics at the University of Michigan.

The mathematical club at the University of Minnesota is taking up the study of the fundamental existence theorems given by Professor G. A. BLISS in the Princeton Colloquium Lectures recently published by the American Mathematical Society.

The January number of *School Science and Mathematics* contains two articles that are of interest to readers of the MONTHLY. They are "Suggestions for the prospective mathematician," by G. A. MILLER, and "Needed,—a funeral of algebraic phraseology," by EFFIE GRAHAM, of the high school at Topeka, Kansas.

An answer-book to the DAVIS-BRENKE Calculus has just been issued by the Macmillan Company. As such an answer-book saves much of the instructor's time, it adds to his efficiency as a teacher.

A review of Paolo Ruffini's researches, carried on during the years 1799–1813, on the impossibility of the algebraic solution of the quintic and on groups of operations, is given by E. BORTOLOTTI in the *Memorie della R. Accad. di Scienze etc. in Modena*, S. 3, Vol. 12 (Sezione scienze). He aims to give some of the results reached by Ruffini in order to remove some misconceptions that are prevalent relating to Ruffini's work and in order to encourage students to read Ruffini's original publications.

Nature (Dec. 18, 1913) contains a review of A. N. WHITEHEAD and B. RUSSEL'S *Principia Mathematica*, Vol. III, 1913. This volume treats of the theory of series and the theory of measurement. As in the preceding volumes, so here the authors use a new symbolism and deal with the logical deduction of the propositions from merely logical foundations.

The engineering departments of Harvard University and Massachusetts Institute of Technology are to be combined, according to a press dispatch, the departments to be conducted in the buildings of the Institute now being erected in Cambridge. The president of the Institute will be the executive head of the work, and the faculty will consist of the faculty of the Institute, enlarged by the addition of Harvard professors in the departments involved. The two institutions are to remain unchanged in name, organization, and title to property, the Institute of Technology to furnish the buildings, laboratories, and the two institutions to furnish jointly equipment and contributions to the necessary funds.

At the November meeting of the Association of Teachers of Mathematics in the Middle States and Maryland the following papers were read: MAURICE J. BABB and CHARLES F. WHEELOCK, "Are particular abilities necessary for the pupil to gain an understanding of the elementary and secondary mathematics as usually given at the present time"; JAMES C. BROWN, "A comparison at equal

school ages of the attainments in mathematics of the European and American schoolboy, with a consideration of causes and remedies"; ALBERT D. YOCUM, "Mathematics as a means to culture and discipline"; ROMIETT STEVENS, "The use of the question in the classroom."

During the recent meeting of the American Association for the Advancement of Science, held at Atlanta, Georgia, Professor H. S. WHITE of Vassar College was elected chairman of Section A. The retiring chairman of this Section, Professor E. B. VAN VLECK of Wisconsin University, delivered an address entitled "The influence of Fourier's series upon the development of mathematics." The next meeting of this Association will be held in Philadelphia, during the convocation week, 1914–15.

According to the *Bulletin des Sciences Mathématiques*, December, 1913, page 355, the Minister of Public Instruction of France began to study the question of requiring "candidats à l'agrégation des sciences mathématiques" to pass an examination in the elements of the theory of groups and their applications in the theory of equations.

During the public meeting of the Paris Academy of Sciences, held December 15, 1913, it was announced that the "Grand Prix" (3,000 francs) for the year 1916 would be given for a work applying the methods of POINCARÉ to the integration of some linear algebraic differential equations. The Academy reserves the right to examine published papers as well as manuscripts in connection with this prize.

At the same meeting of the Paris Academy of Sciences especial attention was called to the important work by M. SUNDMANN, a young astronomer of Finland, towards the solution of the famous problem of three bodies. The Academy awarded him the Pontécoulant prize, the amount of the award being doubled (1,400 francs). The recommendation was made by a commission of which ÉMILE PICARD was chairman. The statement of the problem of three bodies is well-known: To find the paths described, and the velocities at each point, of three bodies which attract one another according to Newton's law, it being supposed that the bodies are reduced to material points devoid of extension. Sundmann's research is along the path first blazed by Painlevé, continued by Levi-Civita and others. It is interesting to recall that in 1772 this Academy awarded a prize for an essay on the problem of three bodies to the illustrious LAGRANGE. An illuminating historical review of the problem of several bodies was given some years ago by President Edgar O. Lovett of the Rice Institute in Houston, Texas. It is published in *Science,* N. S., Vol. 29, 1909, pp. 81–91.

Copies of the January, 1913, issue of the MONTHLY are *very much needed.* Any one having an extra copy of this number will confer a great favor by sending it to the MANAGING EDITOR at once.

THE

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A NOTE ON PLANE KINEMATICS.

By ALEXANDER ZIWET AND PETER FIELD, University of Michigan.

INTRODUCTION.

It appears from the nature of the subject that the use of vector methods must be of advantage in studying the motion of a rigid body. A. FÖPPL'S *Technische Mechanik,* F. CASTELLANO'S *Meccanica razionale* (1894), and especially R. MARCOLONGO'S *Meccanica razionale* (Milano, Hoepli, 1905, 2 vols., German edition by TIMERDING, Teubner, 1911–12) bear evidence to this effect.

The present paper is confined to *plane* motion where the number of new symbols is very small. The notation used is that of C. BURALI-FORTI et R. MARCOLONGO, *Eléments de calcul vectoriel,* Paris, Hermann, 1910.

The vector from the point O to the point P is denoted by $P - O$, or by a single bold-faced letter such as $\boldsymbol{p}$. The time-derivative of a moving point P is denoted by a dot; thus, if P, P' are the positions of the point at the times t, $t + \Delta t$, we have

$$\dot{P} = \lim_{\Delta t=0} \frac{P' - P}{\Delta t}$$

for the *velocity* of P. The *acceleration* of P is the second time-derivative $\ddot{P}$. The time-derivative of the vector $\boldsymbol{p} = P - O$ is $\dot{\boldsymbol{p}} = \dot{P} - \dot{O}$; this is the velocity of P *relative* to O.

The absolute value, or length, of a vector $\boldsymbol{p} = P - O$ is indicated by mod $\boldsymbol{p}$, or mod $(P - O)$. The scalar product of two vectors $\boldsymbol{a}$, $\boldsymbol{b}$, i. e., the product of their lengths into the cosine of the angle of their directions, is written $\boldsymbol{a} \times \boldsymbol{b}$; for $\boldsymbol{a} \times \boldsymbol{a}$ we write $\boldsymbol{a}^2$. The vector product $\boldsymbol{a} \wedge \boldsymbol{b}$ (read $\boldsymbol{a}$ vec $\boldsymbol{b}$), which is the vector at right angles to both $\boldsymbol{a}$ and $\boldsymbol{b}$, representing by its length the area of the parallelogram formed by $\boldsymbol{a}$ and $\boldsymbol{b}$, is not required in the plane.

Rotation through a right angle in the positive (say, counter-clockwise) sense is indicated by i; thus $i\boldsymbol{a}$, $i\dot{P}$ are the vectors obtained by turning $\boldsymbol{a}$, $\dot{P}$ through

the angle $+\frac{1}{2}\pi$. Finally, the vector obtained by turning the vector $\boldsymbol{a}$ through the angle φ is written $e^{\phi i}\boldsymbol{a}$.

These are all the special notations required in what follows.

Most of the results obtained are of course well known; but it is believed that the constructions of Arts. 6, 7, 11–16 are new.

I. Velocities.

1. If O, P are any two different points of the moving figure and we put $P - O = \boldsymbol{p}$, the condition of rigidity $\boldsymbol{p}^2 =$ const. gives upon differentiation with respect to the time:

$$\dot{\boldsymbol{p}} \times \boldsymbol{p} = 0, \quad \text{or} \quad (\dot{P} - \dot{O}) \times (P - O) = 0;$$

i. e., *the relative velocity* $\dot{\boldsymbol{p}} = \dot{P} - \dot{O}$ *of any point with respect to any other point of the figure is perpendicular to the line joining the points* (Fig. 1).

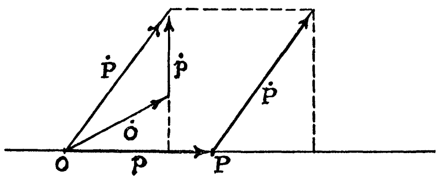


Fig. 1.

In other words, the projections of the velocities $\dot{O}$, $\dot{P}$ of any two points O, P on OP are equal in magnitude and sense.

2. The relation of the last article can be written in the form

$$\dot{\boldsymbol{p}} = wi\boldsymbol{p}, \quad \text{or} \quad \dot{P} - \dot{O} = wi(P - O), \tag{1}$$

where w is a (positive or negative) scalar.

This scalar is independent of the point P. For if Q be any other point of the figure and we put $Q - O = \boldsymbol{q}$ we have the additional conditions of rigidity $\boldsymbol{q}^2 =$ const., $\boldsymbol{p} \times \boldsymbol{q} =$ const., which give upon differentiation:

$$\dot{\boldsymbol{q}} \times \boldsymbol{q} = 0, \qquad \dot{\boldsymbol{p}} \times \boldsymbol{q} + \boldsymbol{p} \times \dot{\boldsymbol{q}} = 0.$$

Writing the former equation in the form $\dot{\boldsymbol{q}} = w_1 i\boldsymbol{q}$ and substituting in the latter equation for $\dot{\boldsymbol{p}}$ and $\dot{\boldsymbol{q}}$ their values we obtain (since $\boldsymbol{a} \times i\boldsymbol{b} = -i\boldsymbol{a} \times \boldsymbol{b}$):

$$(w - w_1) i\boldsymbol{p} \times \boldsymbol{q} = 0.$$

This gives $w_1 = w$ unless $i\boldsymbol{p}$ is perpendicular to $\boldsymbol{q}$, i. e., unless O, P, Q are collinear. But in this case we have $\boldsymbol{q} = m\boldsymbol{p}$ where m is a constant scalar; hence $\dot{\boldsymbol{q}} = m\dot{\boldsymbol{p}}$, which gives again $w_1 = w$.

3. If $w = 0$ it appears from (1) that the velocities of any two points, and hence the velocities of all points, of the figure are equal. This case presents no further interest.

If $w \neq 0$ equation (1) shows that the velocities of all points P relative to any one point O are at right angles to OP and proportional to the distance OP. For this reason the scalar w is called the ANGULAR VELOCITY of the plane motion.

4. If $w \neq 0$ we can find a point of the figure whose velocity is instantaneously zero. For if this point be called C we have from (1) the condition $0-\dot{O}=wi(C-O)$; dividing by w and multiplying by i we find:

$$C = O + \frac{1}{w} i\dot{O}. \tag{2}$$

There exists only one such point. For, dividing (1) by w and multiplying it by i we have

$$\frac{1}{w} i\dot{P} - \frac{1}{w} i\dot{O} = - P + O,$$

whence

$$P + \frac{1}{w} i\dot{P} = O + \frac{1}{w} i\dot{O} = C,$$

by (2); i. e., the same point C is reached from whichever point P of the figure we may start.

This point C of zero velocity is called the INSTANTANEOUS CENTER of the plane motion.

5. If the velocity $\dot{O}$ of any point O and the angular velocity w are given, the instantaneous center C is found, according to (2), by turning $\dot{O}$ about O through $\pi/2$ and multiplying it by $1/w$.

It follows that C lies on the normal of the path of every point of the figure and hence can also be found as the intersection of the normals to the paths of any two points whose velocities are not parallel (Fig. 2).

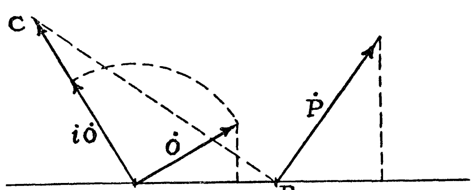


FIG. 2.

If in (1) we take for the arbitrary point O the instantaneous center C whose velocity is zero we find:

$$\dot{P} = wi(P - C); \tag{3}$$

i. e., the velocity of every point P of the figure is perpendicular to its radius vector drawn from C and proportional to its distance from C; in other words, *if not a translation* (all velocities equal, $w = 0$), *the instantaneous motion of a plane rigid figure in its plane is a rotation about the instantaneous center, with angular velocity* w.

6. If the velocities $\dot{O}$, $\dot{P}$ of any two points O, P of the figure are given (subject to the condition of Art. 1), the velocity $\dot{Q}$ of any other point Q of the figure can be found without constructing the instantaneous center. For, the fundamental relation (1) shows that if the vectors $\dot{O}$, $\dot{P}$ be drawn from the same origin, say O (Fig. 3), the line-segment $O'P'$ joining their extremities is perpendicular to OP and w times OP. It follows that the extremities of the velocities of any number of points of the figure, if drawn from the same origin, will form a figure similar to that of the points (though turned through a right angle), the ratio of similarity being w.

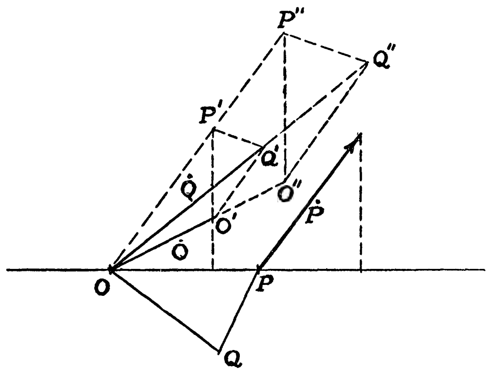


FIG. 3.

Hence, to find $\dot{Q}$ from $\dot{O}$ and $\dot{P}$ make (Fig. 3) $O' - O = \dot{O}$, $P' - O = \dot{P}$, and construct on $O'P'$ a triangle $O'P'Q'$ similar to OPQ, by drawing $O'Q'$ at right angles to OQ and $P'Q'$ at right angles to PQ; then $Q' - O = \dot{Q}$.

If we had used for the velocities a unit $= 1/w$ times the unit of length, making (Fig. 3) $O'' - O = (1/w)\dot{O}$, $P'' - O = (1/w)\dot{P}$, we should have $O''P'' = OP$ and triangle $O''P''Q''$ congruent to triangle OPQ; O is the center of similarity of the triangles $O'P'Q'$ and $O''P''Q''$.

7. Another method for deriving $\dot{Q}$ from $\dot{O}$ and $\dot{P}$ consists in using projections, according to the last statement of Art. 1. The projections of the required velocity $\dot{Q}$ on OQ and PQ must be equal, respectively, to the projections of $\dot{O}$ on OQ and

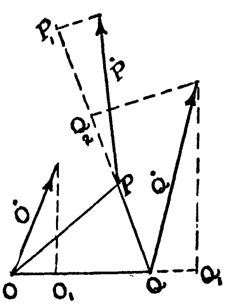


FIG. 4.

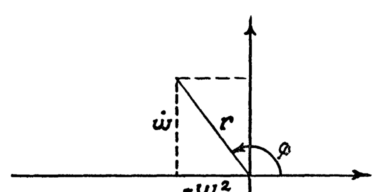


FIG. 5.

of P on PQ; and from these known projections Q can be found (Fig. 4): make $QQ_1 = OO_1$, $QQ_2 = PP_1$, and draw through Q_1 a perpendicular to OQ, through Q_2 a perpendicular to PQ.

II. Accelerations.

8. The fundamental relation (1) gives upon differentiation with respect to the time:

$$\ddot{\boldsymbol{p}} = wi\dot{\boldsymbol{p}} + \dot{w}i\boldsymbol{p} = -w^2\boldsymbol{p} + \dot{w}i\boldsymbol{p}, \tag{4}$$

i. e.,

$$\ddot{P} = \ddot{O} + (-w^2 + \dot{w}i)(P - O).$$

Let us write the complex number $-w^2 + \dot{w}i$ in the polar form (Fig. 5):

$$-w^2 + \dot{w}i = re^{\varphi i},$$

so that

$$r = \sqrt{w^4 + \dot{w}^2}, \qquad \tan\varphi = \frac{\dot{w}}{-w^2}.$$

Taking r positive we have $\frac{1}{2}\pi \leqq \varphi \leqq \frac{3}{2}\pi$ and $\varphi \lesseqgtr \pi$ according as $\dot{w} \gtreqless 0$. The case of translation when both w and $\dot{w}$ are zero is here excluded.

We then have simply

$$\ddot{\boldsymbol{p}} = re^{\varphi i}\boldsymbol{p}, \quad \text{or} \quad \ddot{P} = \ddot{O} + re^{\varphi i}(P - O); \tag{4'}$$

i. e., *the acceleration $\ddot{P}$ of any point P is found from the acceleration $\ddot{O}$ of any other point O by adding to $\ddot{O}$ the vector obtained by turning $P - O$ through the angle φ and multiplying it by r*, φ and r being, at a given instant, the same for all points P of the figure.

9. As $r \neq 0$ (the case of translation being excluded) we can always find one and only one point, say A, whose acceleration is zero. For, the condition $0 = \ddot{O} + re^{\varphi i}(A - O)$ gives

$$A = O - \frac{1}{r}e^{-\varphi i}\ddot{O}. \tag{5}$$

This point A of zero acceleration is called the CENTER OF ACCELERATION.

If in the general relation (4′) we select for O the center of acceleration we find

$$\ddot{P} = re^{\varphi i}(P - A); \tag{6}$$

i. e., *the acceleration $\ddot{P}$ of any point P is obtained by turning the radius vector of P, drawn from A, through the angle φ and multiplying it by r.*

The acceleration $\ddot{P}$ has therefore two components: one, $w^2 \bmod (P - A)$, directed toward A, the other, $\dot{w} \bmod (P - A)$, at right angles to AP.

All points of a circle about A have accelerations of equal magnitude and equally inclined to their radii vectores drawn from A; all points of a line through A have parallel accelerations, in magnitude proportional to the distance from A.

10. If, in particular, $\dot{w} = 0$ we have (Art. 8) $r = w^2$, $\varphi = \pi$, so that (5) and (6) reduce to

$$A = O + \frac{1}{w^2}\ddot{O}, \qquad \ddot{P} = -w^2(P - A).$$

Thus, *if the angular velocity w is constant (but $\neq 0$), the accelerations of all points are directed toward one and the same point, the center of acceleration, and are proportional to the distance from this point.*

If, on the other hand, $w = 0$ while $\dot{w} \neq 0$, we have if $\dot{w}$ is positive: $r = \dot{w}$, $\varphi = \frac{1}{2}\pi$, and if $\dot{w}$ is negative: $r = -\dot{w}$, $\varphi = \frac{3}{2}\pi$, so that (5) and (6) reduce to

$$A = O + \frac{1}{\dot{w}}i\ddot{O}, \qquad \ddot{P} = \dot{w}i(P - A).$$

Thus, *if the angular velocity w is zero (while $\dot{w} \neq 0$), the accelerations of all points are at right angles to their radii vectores drawn from the center of acceleration and proportional to the distance from it.*

11. Comparing the equations (1) and (4′) it appears that the field of acceleration vectors, like that of velocity vectors, is completely determined by any two vectors. But in the case of the velocities the vectors $\dot{O}$, $\dot{P}$ cannot be selected arbitrarily since (Art. 1) $\dot{\mathbf{p}} \times \mathbf{p} = 0$, i. e., $\dot{O}$, $\dot{P}$ must have equal projections on OP. In the case of the accelerations the vectors $\ddot{O}$, $\ddot{P}$ can be taken arbitrarily.

Multiplying (4) by $\times\mathbf{p}$ we find

$$\ddot{\mathbf{p}} \times \mathbf{p} = -w^2\mathbf{p}^2; \tag{7}$$

i. e., *the projections of the accelerations $\ddot{O}$, $\ddot{P}$ of any two points O, P of the figure on the line OP differ by a quantity (w^2 mod $\mathbf{p}$) proportional to the distance OP of the points.*

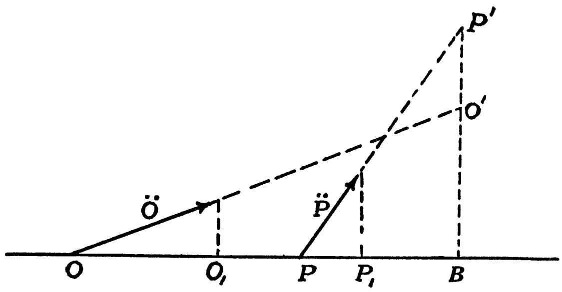


FIG. 6.

To exhibit this relation graphically let us take the unit of acceleration as $1/w^2$ times the unit of length; the difference of the projections of the acceleration vectors, drawn on this scale, will then be equal to the distance OP; i. e., if these vectors be drawn from O and P, the projections of their extremities on OP will coincide, say at B.

Thus, if OO_1, PP_1 (Fig. 6) are the projections on OP of the actual accelerations

(drawn on the same scale as the distances), and if $O' - O = (1/w^2)\ddot{O}$, $P' - P = (1/w^2)\ddot{P}$, the projections of O' and P' on OP coincide at B. As OB is divided by O_1 in the same ratio (viz., w^2) in which PB is divided by P_1, we have

$$\frac{OB}{OO_1} = \frac{OB - OP}{PP_1}, \qquad \text{whence} \qquad OB = \frac{OO_1}{OO_1 - PP_1} \cdot OP,$$

or putting $PP_1/OO_1 = m$:

$$OB = \frac{1}{1 - m} OP.$$

12. This leads to a simple *construction of the center of acceleration A*. Indeed, A is the intersection (different from B) of the circles described about OO' and PP' as diameters (Fig. 7).

For, any point Q on the circle about OO' has its acceleration at right angles to OQ since the projection of OO' on OQ is OQ; and similarly any point R of the circle about PP' has its acceleration at right angles to PR. Hence the acceleration of A, which would have to be perpendicular to both OA and PA, must be zero.

13. The acceleration of O, if represented by $O' - O$, has the rectangular

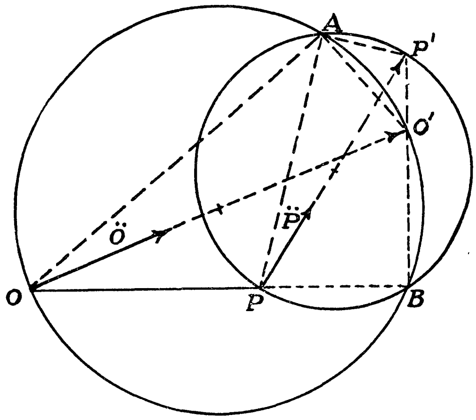


FIG. 7.

components $A - O$ and $O' - A$; similarly the acceleration $P' - P$ of P has the rectangular components $A - P$ and $P' - A$. Now (Fig. 7) $\sphericalangle OO'A = OBA$, $\sphericalangle OBA = PBA = PP'A$; hence $\sphericalangle OO'A = PP'A$. The rightangled triangles OAO', PAP' are therefore similar, and hence

$$\frac{OA}{AO'} = \frac{PA}{AP'}.$$

This means that if the acceleration of any point be resolved along and at right

angles to the radius vector drawn from the center of acceleration, these components have the same ratio for all points of the figure. This, of course, also follows from formula (6) since $\measuredangle AOO'$ is the supplement of φ.

The above construction of the center of acceleration follows also from formula (5) which can be written:

$$A = O + \frac{1}{r} e^{(\pi-\phi)i}\ddot{O} = O + \frac{w^2}{r} e^{(\pi-\phi)i} \cdot \frac{1}{w^2}\ddot{O},$$

or since $w^2/r = -\cos\varphi$(Art. 8):

$$A = O - \cos\varphi \cdot e^{(\pi-\phi)i} \cdot \frac{1}{w^2}\ddot{O}.$$

14. It follows from the preceding articles that *if the accelerations $\ddot{O}$, $\ddot{P}$ of any two points O, P of the figure are given, the acceleration $\ddot{Q}$ of any other point Q can be found,* provided $w \neq 0$. For, by Art. 11 we can find the point B, and from this by Art. 12 the center of acceleration A; hence, constructing on AQ a triangle AQQ' similar to AOO' we find

$$\frac{1}{w^2}\ddot{Q} = Q' - Q, \quad \text{where} \quad w^2 = \frac{\operatorname{mod} \ddot{O}}{\operatorname{mod} (O' - O)}.$$

This construction cannot be used when $w = 0$. But in this case (Art. 10) A is found as the intersection of perpendiculars through O, P to $\ddot{O}$, $\ddot{P}$, respectively. If these perpendiculars happen to be parallel, A lies at infinity; if they coincide, A is found as the intersection of OP with the line joining the extremities of $\ddot{O}$ and $\ddot{P}$.

15. The acceleration $\ddot{Q}$ of any point Q of the figure can however be found from the accelerations $\ddot{O}$, $\ddot{P}$ of any two points O, P without first constructing the center of acceleration. The construction follows directly from the relations $\ddot{\boldsymbol{p}} = re^{\phi i}\boldsymbol{p}$, $\ddot{\boldsymbol{q}} = re^{\phi i}\boldsymbol{q}$, where $\boldsymbol{q} = Q - O$ (Fig. 8). The vectors $P - O = \boldsymbol{p}$ and $\ddot{P} - \ddot{O} = \ddot{\boldsymbol{p}}$,

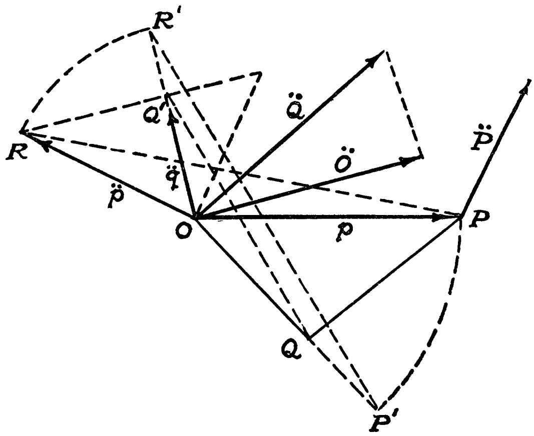


FIG. 8.

drawn from O, form a triangle OPR with the angle φ at O. Turn this triangle about O through an angle $= \measuredangle POQ$ so that it takes the position $OP'R'$; a parallel through Q to $P'R'$ then determines $\ddot{q} = Q' - O$. Adding this to $\ddot{O}$ we find $\ddot{Q}$.

16. Finally, by using projections somewhat as in Art. 7 for velocities, we can derive $\ddot{Q}$ from $\ddot{O}$ and $\ddot{P}$ as follows.

Let OO_1 and OP_1 be the projections on OP of $\ddot{O}$ and $\ddot{P}$, drawn from O (Fig. 9), and construct on P_1O_1 the triangle $P_1O_1Q_1$ similar to OPQ. Let the parallel to OP through Q_1 meet OQ at Q', PQ at Q''; and let OO' be the projection of $\ddot{O}$ on OQ, PP' the projection of $\ddot{P}$ on PQ.

Then $Q'O'$ is the projection of $\ddot{O}$ on QO, $Q''P'$ the projection of $\ddot{Q}$ on QP; transferring these to Q, $\ddot{Q}$ is found.

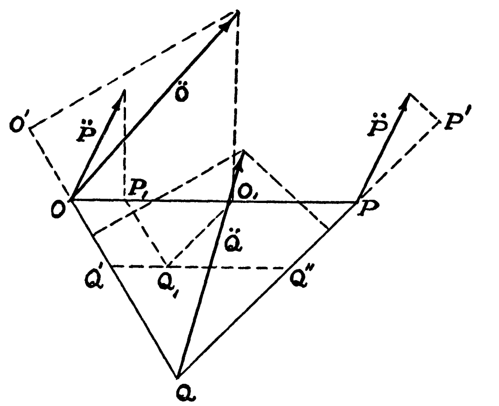


FIG. 9.

The constructions of Arts. 7, 11–13, 16 were first obtained from the equations in Cartesian coordinates, OP being taken as axis of x. Using the vector method it is naturally preferable to deal with the vectors themselves rather than with their projections.

NOTE ON CERTAIN ALGEBRAIC EQUATIONS.

By HERMON L. SLOBIN, University of Minnesota.

In elementary analysis we often meet problems like the following:

Show that the roots of the equation

I. $$x^3 + x^2 - 2x - 1 = 0 \quad \text{are} \quad 2\cos\frac{2}{7}\pi, \quad 2\cos\frac{4}{7}\pi, \quad 2\cos\frac{8}{7}\pi.$$

Show that the roots of the equation

II. $$x^3 - 5x^2 + 6x - 1 = 0 \quad \text{are} \quad 4\cos^2\frac{\pi}{7}, \quad 4\cos^2\frac{2\pi}{7}, \quad 4\cos^2\frac{4\pi}{7}.$$

These problems are taken from Bromwich, "Theory of Infinite Series," page 188. The method of solution used by Bromwich involves a knowledge of certain formulæ in the theory of trigonometric series. These formulæ are based upon the consideration of a certain differential equation, and the method of solution by their means is applicable only when these roots of the algebraic equations have very special forms.

Problems may be stated which require algebraic equations to be set up whose roots are given trigonometric forms such as the above. In a previous paper[1] it has been shown that the trigonometric functions whose arguments are rational multiples of π, are algebraic numbers. The general problem would then be to find the algebraic equation whose roots are $\theta_1(r_1\pi)$, $\theta_2(r_2\pi)$, $\cdots$ $\theta_n(r_n\pi)$, where θ_i represents any trigonometric function, $i = 1, 2, \cdots n$, and r_i represents any rational number. More generally, θ_i may represent any expression built up of the trigonometric functions whose arguments are rational multiples of π, by means of the operations of addition, subtraction, multiplication, division, raising to powers, and extracting roots.

A formal elementary method is readily obtainable for the solution of such problems, but in the general case the operations of reducing the resultant equation to an algebraic equation with rational coefficients would be quite cumbersome, and the degree of the final equation often much larger than the number of roots assigned as such trigonometric forms. However, in particular cases, where the degree of the rational algebraic equation is the same as the number of roots assigned as trigonometric forms, the operations are simple, and the *process* is applicable even in the general case.

I shall illustrate this method by applying it to the problems stated above. The detailed operations are instructive and the forms in which the coefficients enter in the equation are of especial interest.

1. Given the roots $a_1 = 2\cos\dfrac{2\pi}{7}$, $a_2 = 2\cos\dfrac{4\pi}{7}$, $a_3 = 2\cos\dfrac{8\pi}{7}$.

From De Moivre's theorem, $(\cos x + i\sin x)^r = \cos rx + i\sin rx$, when $x = \pi$, and r is any rational number, we have

$$\cos r\pi = \frac{(-1)^{2r} + 1}{2(-1)^r},$$

where $(-1)^r$ denotes *any of the complex roots*. That is, if $r = p/q$, $(-1)^{p/q}$ means the pth power of any of the complex qth roots of -1. Hence

$$a_1 = \frac{(-1)^{\frac{4}{7}} + 1}{(-1)^{\frac{2}{7}}}, \quad a_2 = \frac{(-1)^{\frac{8}{7}} + 1}{(-1)^{\frac{4}{7}}}, \quad a_3 = \frac{(-1)^{\frac{16}{7}} + 1}{(-1)^{\frac{8}{7}}}.$$

The equation

$$x^3 - (a_1 + a_2 + a_3)x^2 + (a_1a_2 + a_1a_3 + a_2a_3)x - a_1a_2a_3 = 0$$

will now be constructed.

[1] Bauer and Slobin, *Rendiconti*, 1913.

Multiplying numerator and denominator of a_1 by $(-1)^{\frac{5}{7}}$ and reducing the numerator by substituting (-1) for $(-1)^{\frac{7}{7}}$, we have

$$a_1 = (-1)^{\frac{2}{7}} - (-1)^{\frac{5}{7}};$$

similarly

$$a_2 = (-1)^{\frac{4}{7}} - (-1)^{\frac{3}{7}};$$
$$a_3 = -(-1)^{\frac{1}{7}} + (-1)^{\frac{6}{7}}.$$

Hence

$$a_1 + a_2 + a_3 = [(-1)^{\frac{6}{7}} - (-1)^{\frac{5}{7}} + (-1)^{\frac{4}{7}} - (-1)^{\frac{3}{7}} + (-1)^{\frac{2}{7}} - (-1)^{\frac{1}{7}}].$$

We will denote this bracket by K. From $x^7 + 1 = 0$ we have for all the complex roots of $(-1)^{\frac{1}{7}}$,

$$x^6 - x^5 + x^4 - x^3 + x^2 - x = -1.$$

Hence

$$K = -1.$$

Likewise

$$a_1a_2 = (-1)^{\frac{6}{7}} + (-1)^{\frac{2}{7}} - (-1)^{\frac{5}{7}} - (-1)^{\frac{1}{7}},$$
$$a_1a_3 = -(-1)^{\frac{3}{7}} + (-1)^{\frac{6}{7}} - (-1)^{\frac{1}{7}} + (-1)^{\frac{4}{7}},$$
$$a_2a_3 = -(-1)^{\frac{5}{7}} + (-1)^{\frac{4}{7}} - (-1)^{\frac{3}{7}} + (-1)^{\frac{2}{7}}.$$

Hence

$$a_1a_2 + a_1a_3 + a_2a_3 = 2K = -2,$$

and finally

$$a_1a_2a_3 = 2 + K = 1.$$

Therefore the required equation is

$$x^3 + x^2 - 2x - 1 = 0.$$

2. Given the roots $a_1 = 4\cos^2\frac{\pi}{7}$, $a_2 = 4\cos^2\frac{2\pi}{7}$, $a_3 = 4\cos^2\frac{4\pi}{7}$. By a process similar to the above, we have

$$a_1 = (-1)^{\frac{2}{7}} - (-1)^{\frac{5}{7}} + 2,$$
$$a_2 = (-1)^{\frac{4}{7}} - (-1)^{\frac{3}{7}} + 2.$$
$$a_3 = -(-1)^{\frac{1}{7}} + (-1)^{\frac{6}{7}} + 2.$$

Also,

$$a_1 + a_2 + a_3 = 6 + K = 5,$$
$$a_1a_2 + a_1a_3 + a_2a_3 = 12 + 6K = 6,$$
$$a_1a_2a_3 = 10 + 9K = 1.$$

Hence the equation is

$$x^3 - 5x^2 + 6x - 1 = 0.$$

As a further example a case involving somewhat more difficult rationalizations is proposed by the author in the problem department in this issue, namely, to find the algebraic equation whose roots are

$$a_1 = \cos\frac{\pi}{9},\ a_2 = -\cos\frac{2\pi}{9},\ a_3 = -\cos\frac{4\pi}{9}.$$

A PLEA FOR LESS FORMAL WORK IN MATHEMATICS.

By F. M. MORGAN, Dartmouth College.

An American professor of mathematics while abroad was told by one of his English colleagues, that American mathematical text-books looked as if they had been written for the feeble-minded. The American professor replied: "Perhaps that is so but your text-books look as if they had been written by the feeble-minded." Now I feel like Sir Roger de Coverly when he said "Much may be said on both sides." However in this brief paper I intend to discuss the question only from the foreign professor's point of view.

First I ask why should our American texts as a whole call for such a remark? I believe we do not have to look far in order to find the answer. It lies in the fact that the majority of them contain too may rules and formulæ. They are written chiefly with the idea of cramming a student to pass an examination and not with the idea of teaching him mathematics. The stimulus for original thinking is therefore sacrificed, everything being done by a cut and dried rule. Many of our texts make mathematics a subject for memory instead of a subject for reason. They try to feed the student mathematics in the form of little sugar-coated pills.

There are of course certain rules and formulæ which are fundamental to the science. These must be taught to, and memorized by, the student. There are others which are not fundamental and should not be taught, for generally these non-important rules and formulæ hold as prominent a place in the student's mind as the important ones do. Let me illustrate my point by a concrete example. A large number of our American text-books on algebra give the following rule for evaluating a third order determinant: "Rewrite the first and second columns to the right of the determinant. The diagonals running down from left to right give the positive terms. The diagonals running down from right to left give the negative terms." Now this rule is so simple that any child can use it, but I ask, of what real good is the rule? It does not apply to a determinant of any other order, and therefore it is not a rule fundamental to the study of determinants. Expansion by means of minors is just as easy, and it is, moreover, fundamental to the subject since it may be applied to a determinant of any order.

Then we have rules in which the emphasis on the fundamental principle is sacrificed in order to make the rule short so that a student without mental effort may solve a problem of that type. Let me also illustrate this point by means of the usual rule for finding the equation of a straight line through a given point and parallel to a given line: "*Step* 1. Change the constant term in the given equation to k. *Step* 2. Substitute the coördinates of the given point in the equation of step 1 and solve for k. *Step* 3. Substitute this value of k in the equation of step 1, and the result is the desired equation." This rule, like the one on determinants, is easy to learn and apply, but what real knowledge of analytic geometry has the student gained by learning it? How many students

really know why the above rule gives the desired result and not, for example, the equation of a line through the given point and perpendicular to the given line? Very few, it will be found, really know. The reason for this is that the fundamental principle involved, namely that parallel lines have the same slope but different intercepts on the y axis, has been mentioned implicitly and not explicitly. If such rules as the above (and there are many more of the same type) were written so that the fundamental principle involved were explicitly stated, the student would benefit greatly thereby.

Experience makes it certain that a student learns more mathematics, can pass a better examination (if that is a criterion for excellency in the subject) if he knows the fundamental formulæ, the fundamental principles, and has been taught to reason and not memorize.

BOOK REVIEWS.

Edited by W. H. Bussey.

Advanced Algebra. By Joseph V. Collins. American Book Company, New York. x + 342 pages. $1.00.

The subject matter for a freshman course in algebra as presented in the great majority of text books does not vary much. This book differs slightly from the usual algebra in this respect. It is divided into three parts. Part I, containing five chapters, is devoted to a review of the elementary algebra through simultaneous quadratics. At the end of chapter four is a list of the fundamental principles. This is followed by a short explanation of the most common errors arising from not following these principles. The fifth chapter contains a list of the theorems of plane geometry used in the rest of the book.

Part II is made up of the chapters on graphs, ratio and proportion, logarithms, the progressions, annuities, the binomial theorem, and inequalities. The author makes evident the connection between algebra and trigonometry by following proportion with the definitions of sine, cosine, tangent and cotangent, and applying these to problems involving right triangles.

In Part III are found the topics which belong to advanced algebra proper; namely, theory of equations, permutations and combinations, probability, determinants, series, undetermined coefficients, continued fractions, and complex numbers. Some of these topics are discussed very briefly. The student is supposed to get many of the principal facts of the subject from simple, concise proofs, yet he is not burdened with too much theory of the kind which does not appeal to the average freshman. In dealing with the complex number, the addition theorem of trigonometry is derived for use in establishing De Moivre's theorem. A further connection between algebra and trigonometry is thus made.

At various places short historical notes are added. These consist, for the most part, of biographical sketches of great mathematicians, and are so placed

that they follow the subject with which the man's name is intimately associated. The chapter on logarithms ends with short notes on Briggs and Napier, that on the theory of equations with one on Gauss, determinants with one on Sylvester. A few pictures of mathematicians are also introduced.

GEO. W. HARTWELL.

Theory of Functions of a Complex Variable. BY DR. H. BURKHARDT, Professor in the Technical School, Munich. Authorized English translation, with the addition of figures and exercises, by S. E. RASOR, Professor of Mathematics, Ohio State University. D. C. Heath & Co., Boston, 1913. 421 pages. $4.00.

This book is a close translation of the second part of the first volume of Professor Burkhardt's *Funktionen-Theorie.* Since the latter is no doubt well known to those interested in this field, it will perhaps be sufficient here to give the chapter headings and to indicate briefly the additions made by the translator. Chapters, sections, and theorems are numbered exactly as in the German text.

Chapter I. *Complex numbers and their geometrical representation.* The translator has added at the end of the chapter 32 exercises designed to give practice in the use of the complex number.

Chapter II. *Rational functions and conformal representations determined by them.* Nine lists of exercises, containing 78 individual problems, have been added. Also, following § 21 there has been added § 21*a*, 7 pages, devoted to the function $w = \frac{1}{2}(z + z^{-1})$; and following § 22 there has been added § 22*a*, 6 pages, treating the function $w = z^3 - 3z$.

Chapter III. *Theory of real variables and their functions.* Four lists containing 31 exercises have been added.

Chapter IV. *Single-valued analytic functions.* A brief section, § 30*a*, on limits of convergent sequences of complex numbers, and ten lists containing 119 exercises have been added by the translator.

Chapter V. *Many-valued analytic functions.* The translator has added § 57*a* on the function $\tan^{-1} z$, § 60*a* on rational functions of z and $\sqrt{z}$, § 62*a* on rational functions of z and $\sqrt{(z - a)(z - b)}$, § 62*b* on integrals of rational functions of z, etc., § 62*c* on the function $z = w + i\sqrt{1 - w^2}$, § 62*d* on the function $\sin^{-1} w$, 27 pages in all, and six lists containing 67 exercises.

Chapter VI. *General theory of functions.* Three lists containing 48 exercises have been added.

This is without doubt a most timely book, for the need of a text in English of about this scope has long been felt. It is to be regretted that some discussion of the logarithmic potential and of streamings in general was not added for the benefit of students of physics who study the theory of functions for its applications. However, with nearly four hundred exercises added to round out in many important particulars the excellent work of Burkhardt, we now have available in English a most satisfactory text-book on the theory of functions of a complex variable.

W. C. BRENKE.

The Principles of Projective Geometry. By J. L. S. HATTON. Cambridge University Press, 1913. x + 366 pages. $3.50.

In the preface to this book the author asserts that the study of pure geometry is at present neglected and expresses the opinion that this is a misfortune. The book has been written with the hope that it may do something to alter this situation and with confidence that "the great landmarks of projective geometry" may be presented to the student in such a manner that real enthusiasm for this branch of mathematics may be aroused. The author has succeeded admirably in his attempt to write a text which gives a clear and interesting exposition of the principal methods and more important theorems of projective geometry.

The mathematical preparation with which a student is equipped when he begins the study of projective geometry is usually based on metrical concepts. Consequently the first difficulties he encounters are those arising from his being introduced all at once into an entirely new geometry from which the idea of measurement has been almost altogether excluded. The author has been aware of these difficulties and has obviated them by leading the student gradually into the non-metric geometry by the aid of metric properties. Thus the student becomes acquainted with anharmonic ratio early in the book; harmonic ratio is first defined as a special case of anharmonic ratio, and its non-metric properties and the construction of a harmonic range (pencil) from the harmonic property of a complete quadrangle (quadrilateral) are given later; the definition of involution based on its metric property is given before the non-metric definition. However, the second definition is given immediately after the first and it is proved that the metric definition might have been deduced from the second. Such a gradual transition into the study of projective geometry seems to be the best method of approach for an introductory course. The discussions and proofs are clearly presented, though enough is usually omitted to require the student to do considerable thinking for himself. The value of the book as a text is greatly enhanced by the many well-selected problems which it contains.

The first three chapters are devoted to the definitions of such fundamental concepts as range of points, pencil of lines, projection, perspective, projective relationships, anharmonic ratio and harmonic ratio. The principle of duality is here discussed and proofs of several of the properties of anharmonic and harmonic ratios, and of Ceva's and Menelaus's theorems are given. The author has used good judgment in these introductory chapters both as to the selection and the arrangement of the material.

Chapter IV contains a vivid exposition of conical (central) projection and of plane perspective. To show more clearly the differences and the similarities existing between conical projection and plane perspective, the constructions of corresponding figures and the proofs of the fundamental theorems of the two have been placed side by side.

In the next four chapters the following topics are taken up: applications of conical projection and of plane perspective to the proofs of a number of interesting theorems on triangles, quadrangles and quadrilaterals; applications of harmonic

ratio to the study of ranges and pencils in perspective, and of projective and superposed ranges and pencils; the harmonic properties of the complete quadrangle and quadrilateral; properties of involution ranges and pencils, particularly those properties of involution which are used in subsequent chapters on the conic.

In order that the matter may be more readily grasped by the student, the relations of projective forms, anharmonic properties, pole and polar, Carnot's, Pascal's and Desargues's theorems, to the circle are studied before these relations are taken up for the general conic. In the chapters on the conic some of the theorems already proved for the circle are simply restated, others are proved by methods similar to those used for circles, while in several cases alternative proofs —such as proofs by projection—are given. These chapters also contain the classification of conics and discussions of center, foci, diameters and asymptotes. The remaining eight chapters of the book deal for the most part with applications of anharmonic and harmonic ratios, involution, pole and polar, Carnot's, Pascal's and Desargues's theorems to further deductions of properties of triangles, of quadrangles and quadrilaterals, and of conics. The large number of applications and problems contained in these chapters cannot fail to interest the student of mathematics.

The addendum contains fifteen theorems and their proofs on non-projective properties of the straight line and circle. There are two very complete indexes, one of terms and definitions and the other of theorems.

The typographical errors are few and of such obvious nature as to cause the student no difficulty. The type is large and distinct and the figures have been carefully constructed.

It is the opinion of the reviewer that this book is better adapted for use as a text in a first course on projective geometry than any other book in the English language. It may also be used for reference in a lecture course on the subject. The book should do much to revive interest in the study of projective geometry in the undergraduate curriculum and "to encourage the student not to neglect the methods of pure geometry."

C. E. STROMQUIST.

PROBLEMS AND SOLUTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

Special Notice.—Please reread the requests as to form of solutions on pp. 258–259 of the October 1913 issue. Unless these directions are observed by contributors, solutions must either be entirely rewritten by the committee or else rejected. Put all drawings on separate sheets.

MANAGING EDITOR.

ALGEBRA.

When this issue was made up, solutions had been received for 401, 402, 403, 404, 405, and 407. A solution of 406 is desired.

408. Proposed by EMMA M. GIBSON, Drury College.

Show that if n is a positive integer, the sum of the series

$$1 - \frac{2n-1}{1!} + \frac{(2n-1)(2n-2)}{2!} - \cdots + (-1)^{n-1}\frac{(2n-1)(2n-2)\cdots(n+1)}{(n-1)!}$$

is

$$\frac{(-1)^{n-1}(2n-2)(2n-3)\cdots(n+1)n}{(n-1)!}.$$

[From C. Smith's *Treatise on Algebra.*]

409. Proposed by C. E. GITHENS, Wheeling, W. Va.

Find integral values for the edges of a rectangular parallelopiped so that its diagonal shall be rational.

410. Proposed by C. N. SCHMALL, New York City.

Solve the simultaneous equations,

$$x^2 + xy + y^2 = a,$$

$$x^4 + x^2y^2 + y^4 = b.$$

411. Proposed by V. M. SPUNAR, Chicago, Illinois.

Determine $x_1, x_2, x_3, \cdots x_p$, from the equations

$$\begin{aligned} x_1 + x_2 + x_3 + \cdots + x_p &= a_0 \\ b_1x_1 + b_2x_2 + b_3x_3 + \cdots + b_px_p &= a_1 \\ b_1^2x_1 + b_2^2x_2 + b_3^2x_3 + \cdots + b_p^2x_p &= a_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ b_1^{p-1}x_1 + b_2^{p-1}x_2 + b_3^{p-1}x_3 + \cdots + b_p^{p-1}x_p &= a_{p-1}. \end{aligned}$$

412. Proposed by H. L. SLOBIN, University of Minnesota.

Form the algebraic equation whose roots are: $a_1 = \cos\frac{\pi}{9}$, $a_2 = -\cos\frac{2\pi}{9}$, $a_3 = -\cos\frac{4\pi}{9}$. (See article in this issue, page 113.)

GEOMETRY.

When this issue was made up, solutions had been received for 431, 434 and 435. Solutions of 430, 432 and 436 are desired.

Those interested in 427 may add to the footnote attached thereto in the January issue the following reference: *Das Problem der Kreisteilung*, von A. Mitzscherling, Leipzig and Berlin, 1913.

437. Proposed by J. BROOKS SMITH, Hampden-Sidney, Va.

Let D, E, F be three arbitrary points taken on the sides of a triangle ABC. If Δ and Δ' be the areas of the triangles ABC and DEF, show that

$$\frac{\Delta'}{\Delta} = \frac{AF\cdot BD\cdot CE + AE\cdot CD\cdot BF}{abc},$$

the sign of each factor being determined as follows: Each segment adjacent to one of the vertices of the triangle ABC is to be regarded as positive or negative according as it is drawn towards or from the other vertex on the side containing the segment.

438. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

By means of the theorem that the product of the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the two products of pairs of opposite sides, obtain the usual formulæ for $\sin(\alpha \pm \beta)$ and $\cos(\alpha \pm \beta)$ in terms of $\sin\alpha$, $\sin\beta$, $\cos\alpha$, $\cos\beta$ (Godfrey and Siddon's *Geometry*, page 82).

439. Proposed by CHARLES N. SCHMALL, New York City.

Show that the areas of any two triangles circumscribed about the same circle are in the same ratio as their perimeters.

440. Proposed by W. L. WATSON, Moundsville, West Va.

A solid sector is cut out of a sphere 10 feet in radius by a cone whose vertical angle is 120°. Find the radius of the sphere whose volume is equal to that of the sector.

CALCULUS.

When this issue was made up, solutions had been received for 344-5-6-7, 349, 351, 354, 356, 357, and 358. Solutions of 332, 337, 340, 342, 348, 350, 352, and 353 are desired. A complete solution of 339 is also desired.

359. Proposed by W. D. CAIRNS, Oberlin College.

Examine for maxima and minima

$$f(x) = e^{-cx}(1 + \cos x) \quad (c > 0).$$

360. Proposed by ELMER SCHUYLER, Brooklyn, New York.

What interpretation must be given to

$$\frac{d^{1/2}y}{dx^{1/2}} \text{ so that } \frac{d^{1/2}}{dx^{1/2}}\left(\frac{d^{1/2}y}{dx^{1/2}}\right) = \frac{dy}{dx}?$$

361. Proposed by EMMA M. GIBSON, Drury College.

Determine the system of curves satisfying the differential equation

$$[(1 + x^2)^{1/2} + ny]dx + [(1 + y^2)^{1/2} + nx]dy = 0,$$

and show that the curve which passes through the point $(0, n)$ contains as part of itself the conic

$$x^2 + y^2 + 2xy(1 + n^2)^{1/2} = n^2.$$

(From Forsyth's *Differential Equations*, p. 41.)

362. Proposed by C. N. SCHMALL, New York City.

Having given $y^3 - a^2y + axy - x^3 = 0$, show by Maclaurin's theorem that

$$y = -\frac{x^3}{a^2} - \frac{x^4}{a^3} - \frac{x^5}{a^4} - \cdots.$$

MECHANICS.

When this issue was made up, solutions had been received for 273, 284-5, 288, and 289. Solutions of 266, 268, 269, 271, 274-5, 277, 279, and 286 are desired.

290. Proposed by B. F. FINKEL, Drury College.

A fox, pursued by a hound, is running with uniform velocity over a frail arch in the form of a cycloid; the hound stops at a weak point of the arch, then tumbles through and reaches the level ground with a velocity equal to that of the fox. Prove that the fox exerted no normal pressure on the arch at the point where the hound fell through.

(From Walton's *Problems in Theoretical Mechanics*, p. 605.)

291. Proposed by EMMA M. GIBSON, Drury College.

The time of descent, down a rough inclined plane, of a spherical shell which contains a smooth solid sphere of the same material as itself is t_1. The time of descent, down the same plane, of a solid sphere of the same material and radius as the shell is t_2. Determine the thickness of the shell.

From Loudon's *Elementary Theory of Rigid Dynamics*, p. 188.

292. Proposed by C. N. SCHMALL, New York City.

In a bombardment, a battleship directs its fire at a fort standing on a hill whose height is a feet above the sea level. The angle of elevation of the fort is found to be ϕ. If the initial velocity of the projectile is v, show that the fort will *not* be struck if $v < \sqrt{ag(1 + \csc\phi)}$.

NUMBER THEORY.

When this issue was made up, solutions had been received for 200, 203, 206, 207 and 210. Solutions of 189, 191, 192, 196, 202, 204–5, and 208–9 are desired.

211. Proposed by E. T. BELL, Seattle, Washington.

If an odd perfect number exists, the total number of its divisors is a multiple of 2, but not of 4; or, what is the same thing, an odd perfect number must be of the form $p^{2a-1}n^2$, where p is prime and a is odd.

212. Proposed by ELMER SCHUYLER, Brooklyn, New York.

Given any positive integer N greater than 1; to prove that the sum of all the positive integers less than N and relatively prime to N equals $\frac{1}{2}N\cdot\varphi(N)$.

213. Proposed by R. D. CARMICHAEL, Indiana University.

Prove that no relatively prime integers x and y exist such that the difference of their fourth powers is a cube.

214. Proposed by A. J. KEMPNER, University of Illinois.

Let a be a positive integer $\geqq 2$, and let $T(n)$ denote the number of distinct divisors of the positive integer n, including both 1 and n, so that $T(1) = 1$, $T(2) = 2$, $T(3) = 2$, $T(4) = 3$, $\ldots$. Show that

$$\Sigma_{n=1}^{n=\infty} T(n)/a^n = \Sigma_{n=1}^{n=\infty} 1/(a^n - 1).$$

The special case $a = 10$ gives, as is easily seen:

$$9\sum_{n=1}^{n=\infty}\frac{T(n)}{10^n} = \frac{1}{1} + \frac{1}{11} + \frac{1}{111} + \frac{1}{1111} + \cdots.$$

SOLUTIONS OF PROBLEMS.

ALGEBRA.

396. Proposed by H. E. TREFETHEN, Colby College.

Show that $1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots = \sqrt{2}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$.

I. Solution by Horace Olson, Chicago, Illinois.

Let S_1 represent the sum of the series, $x + \frac{x^3}{3} - \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \frac{x^{11}}{11} - \cdots$, of which the first member of the proposed equation is a particular case. Let S_2 represent the sum of the series $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots$. By calculus,

$$S_1 = \int_0^x (1 + x^2 - x^4 - x^6 + x^8 + x^{10} - \cdots)dx = \int_0^x \left(\frac{1 + x^2}{1 + x^4}\right)dx$$

$$= \frac{\tan^{-1}(\sqrt{2}x + 1) + \tan^{-1}(\sqrt{2}x - 1)}{\sqrt{2}}.$$

Also,

$$S_2 = \int_0^x (1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots)dx = \int_0^x \left(\frac{1}{1+x^2}\right) dx = \tan^{-1} x.$$

Letting $x = 1$, we find

$$S_1 = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots = \frac{\tan^{-1}(\sqrt{2}+1) + \tan^{-1}(\sqrt{2}-1)}{\sqrt{2}}$$

$$= \frac{\tan^{-1}(\sqrt{2}+1) + \cot^{-1}(\sqrt{2}+1)}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}}.$$

Similarly, we find for $x = 1$,

$$S_2 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \tan^{-1} 1 = \frac{\pi}{4}.$$

Therefore

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots = \sqrt{2}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right).$$

II. Solution by Hermon L. Slobin, University of Minnesota.

If we develop $f(x) = 1$ as a Fourier Series

$$f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_m \sin mx + \cdots,$$

where

$$a_m = \frac{2}{\pi}\int_0^\pi f(x) \sin mx dx,$$

we note that

$$\int_0^\pi \sin mx dx = \frac{1}{m}(1 - \cos m\pi) = 0 \quad \text{or} \quad \frac{2}{m},$$

according as m is even or odd. Hence,

$$1 = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right).$$

Letting $x = \pi/4$, we have

$$\frac{\pi}{4} = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{3} - \frac{\sqrt{2}}{2}\cdot\frac{1}{5} - \frac{\sqrt{2}}{2}\cdot\frac{1}{7} + \cdots\right) = \frac{\sqrt{2}}{2}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \cdots\right).$$

Therefore, the left member of the proposed identity $= \sqrt{2} \cdot \pi/4$.

But we note that, since

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots,$$

upon putting $x = 1$, we have $\pi/4 = 1 - 1/3 + 1/5 - \cdots$ which is the series in

the right member of the proposed identity. Hence

$$1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\cdots=\sqrt{2}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right).$$

III. Solution by J. W. Campbell, University of Chicago.

Let S = sum of the series,

$$\sin 2\theta-\frac{1}{2}\sin 4\theta+\frac{1}{3}\sin 6\theta-\cdots.$$

Then S is the coefficient of i in $e^{2i\theta}-\frac{1}{2}e^{4i\theta}+\frac{1}{3}e^{6i\theta}-\cdots$, that is, in $\log_e(e^{2i\theta}+1)$, or in $\log_e e^{i\theta}(e^{i\theta}+e^{-i\theta})$, or finally in $i\theta+\log_e(2\cos\theta)$, which is θ itself.

Take $\theta=\pi/8$. Then

$$\frac{\pi}{8}=\frac{1}{\sqrt{2}}-\frac{1}{2}+\frac{1}{3\sqrt{2}}+0-\frac{1}{5\sqrt{2}}+\frac{1}{6}\cdots$$

$$=\frac{1}{\sqrt{2}}\left(1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}\cdots\right)-\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{5}\cdots\right).$$

That is,

$$1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\cdots=\sqrt{2}\left[\frac{\pi}{8}+\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{5}\cdots\right)\right]. \qquad (1)$$

By Gregory's Series (Loney's *Trigonometry*, Part II, § 94),

$$\theta=\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta-\cdots,\quad\left(-\frac{\pi}{4}\leqq\theta\leqq\frac{\pi}{4}\right).$$

Take $\theta=\pi/4$. Then $\pi/4=(1-1/3+1/5-\cdots)$. Hence,

$$\frac{\pi}{8}=\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{5}-\cdots\right). \qquad (2)$$

Therefore, from (1) and (2),

$$1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\cdots=\sqrt{2}\left(1-\frac{1}{3}+\frac{1}{5}-\cdots\right).$$

Also solved by C. E. Horne, A. L. McCarty, A. M. Harding, David F. Kelley, Elmer Schuyler, C. N. Schmall, and J. W. Clawson.

ALGEBRA.

397. Proposed by W. H. BUSSEY, University of Minnesota.

12 oxen are turned into a pasture of $3\frac{1}{3}$ acres and eat all the grass in 4 weeks so that the pasture is bare. 21 oxen are turned into a pasture of 10 acres and eat all the grass in 9 weeks. How many oxen will eat all the grass of 24 acres in just exactly 18 weeks, it being assumed that the grass in all the pastures is at the same height when the oxen are turned in, and that the grass grows at a uniform rate.

I. Solution by Christian Hornung, Tiffin, Ohio.

Let x = number of pounds of grass each ox eats per week,
y = number of pounds of grass on each acre at first,
z = number of pounds of grass that grows per week per acre,
k = the number of oxen required in the last condition.

Then $\frac{10}{3}y + \frac{40}{3}z = 48x$, $10y + 90z = 189x$, and $24y + 432z = 18kx$. Eliminating y, we find $10z = 9x$, and $560z = (30k - 576)x$, from which k is found to be 36, the required number of oxen.

II. Solution by Artemas Martin, Washington, D. C.

In the first case one ox eats 1/4 of $3\frac{1}{3}/12$ or 5/72 of an acre, and 5/18 of the growth of that acre, in one week. In the second case one ox eats 1/9 of 10/21 or 10/189 of an acre, and 10/21 of what grows on an acre, in one week.

Since one ox eats the same quantity of grass in one week in each case, therefore, $10/21 - 5/18 = 25/126$ of the growth of one acre during one week is $5/72 - 10/189 = 25/1512$ of an acre; and $25/1512 \div 25/126 = 1/12$ of an acre, what grows on an acre during one week.

$5/72 + 5/18$ of $1/12 = 5/54$, the part of the original quantity of grass on one acre which one ox eats in one week.

$5/54 \times 18 = 5/3$, the quantity of grass, in acres, one ox will eat in 18 weeks.

$24 + (1/12 \times 24 \times 18) = 60$, the quantity of grass, in acres, to be eaten from 24 acres in 18 weeks; and $60 \div 5/3 = 36$, the number of oxen required to eat it.

For other solutions, see my paper on "The 'Pasturage Problem,'" published in the *Mathematical Magazine*, Vol. I, No. 2 (April, 1882), pp. 17–22; also, No. 3 of same volume, pp. 43–44.

Also solved by Albert N. Nauer, M. E. Graber, G. W. Hartwell, H. C. Feemster, Daniel Kreth, Elmer Schuyler, Horace Olson, S. W. Reaves, P. Peñalver, and J. W. Clawson.

GEOMETRY.

425. Proposed by V. M. SPUNAR, Chicago, Illinois.

Find the ratio of the areas A_1 and A_2 of the parabolas formed by projectiles whose ranges are the same and whose angles of projection are complements of each other.

Solution by H. C. Feemster, York College, York, Nebraska.

Let $gx^2 - 2v^2x \cos \theta_1 \sin \theta_1 + 2yv^2 \cos^2 \theta_1 = 0$ and $gx^2 - 2v^2x \cos \theta_2 \sin \theta_2 + 2yv^2 \cos^2 \theta_2 = 0$ be the two required parabolas, where $\theta_1 + \theta_2 = 90°$, and v is the number of feet per second in the initial velocity. Then

$$A = \int_0^{\frac{2v^2 \sin \theta \cos \theta}{g}} \frac{2v^2x \cos \theta \sin \theta - gx^2}{2v^2 \cos^2 \theta} dx = \frac{2}{3} \frac{v^4 \sin^3 \theta \cos \theta}{g^2},$$

giving the ratio

$$\frac{A_1}{A_2} = \frac{\sin^3 \theta_1 \cos \theta_1}{\cos^3 \theta_2 \sin \theta_2} = \tan^2 \theta_1, \quad \text{since} \quad \theta_2 = \frac{\pi}{2} - \theta_1.$$

This also might have been gotten by dividing the maximum altitude of the first projectile, $(v^2 \sin^2 \theta_1)/2g$, by that of the second projectile, $(v^2 \cos^2 \theta_1)/2g$, as the horizontal distance of the two, $(2v^2 \cos \theta \sin \theta)/g$, is the same.

Also note that if $\theta_1 = \theta_2$, we have a maximum horizontal distance. And further, noting the equation, $y = vt \sin \theta - \frac{1}{2}gt^2$, we find the total time $t = (2v \sin \theta)/g$, and $t_1/t_2 = \tan \theta_1$.

Note. This problem was incorrectly listed under Geometry in the November, 1913, issue. It should have been under Mechanics. EDITORS.

Also solved by RICHARD MORRIS, J. L. RILEY, B. L. LIBBY, C. N. SCHMALL, A. M. HARDING, HORACE OLSON, S. W. REAVES, W. C. EELLS, and J. W. CLAWSON.

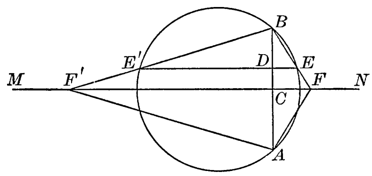
A solution of 424 by J. W. CLAWSON was received too late for publication in the March issue.

426. Proposed by R. D. CARMICHAEL, Indiana University.

On a given chord of a circle as a base construct an isosceles triangle, with vertex outside of the circle, such that its sides shall be divided in a given ratio by their points of intersection with the circle.

SOLUTION BY J. B. SMITH, Hampden-Sidney, Va.

Let AB be the given chord, C its mid-point and MN its perpendicular bisector. Let $m : n$ be the given ratio.



Divide the semi-chord BC in the given ratio. If D be the point of division, erect the perpendicular to AB at D. Let it cut the circle at E, E'; draw BE and produce it to meet MN at F and draw AF. Then AFB is the required triangle. For $AF = BF$ and $BE : EF = BD : DC = m : n$. $BF'A$ is another solution.

Also solved by G. W. HARTWELL, M. E. GRABER, C. HORNUNG, A. M. HARDING, ELMER SCHUYLER, RICHARD MORRIS, KENNETH REYNOLDS, BARNEM LIBBY, J. W. CLAWSON, and EMMA M. GIBSON.

CALCULUS.

341. Proposed by E. B. ESCOTT, University of Michigan.

Find the value for the volume of a barrel in terms of its length l, the bung diameter a and the head diameter b, also an approximate expression when a and b are nearly equal.

I. SOLUTION BY THE PROPOSER.

The simplest curve for the longitudinal cross section of the barrel is probably a parabola. Its equation, since it has its vertex on the y-axis and passes through the points $(-l/2, b/2)$, $(0, a/2)$, $(l/2, b/2)$, is

$$y = \frac{a}{2} - \frac{2(a-b)}{l^2} x^2.$$

The volume of solid obtained by revolving this curve around the x-axis is

$$\pi \int_{-\frac{l}{2}}^{\frac{l}{2}} y^2 dx = \pi \frac{l}{60} (8a^2 + 4ab + 3b^2).$$

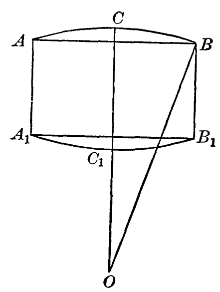
If a and b are nearly equal, $(a - b)^2$ is quite small. So the above expression for the volume may be replaced by

$$\pi \frac{l}{60} [8a^2 + 4ab + 3b^2 + 2(a - b)^2] = \pi \frac{l}{12} (2a^2 + b^2).$$

This latter formula can also be obtained from Simpson's Rule, $V = \frac{l}{6}(S_1 + 4S_2 + S_3)$ where S_1 and S_3 are end sections of the solid and S_2 the mid-section.

II. Solution by A. H. Holmes, Brunswick, Maine.

Let $ACBB_1C_1A_1$ be a longitudinal section of the barrel. Take O as the center of the circumference of a circle of which the arc ACB is a part. Put $OA = r$. The capacity of the barrel is equal to the sum of the contents of the cylinder ABB_1A_1 and the solid of revolution formed by the revolution of the segment ACB about the axis of the barrel. The distance of the center of gravity of the segment ACB from O is $\overline{AB}^3$ divided by 12 times the area of the segment. Therefore,



$$V = \frac{\pi l b^2}{4} + 2\pi \left\{ \frac{l^3}{12} - \left(r - \frac{a}{2}\right)\left[r^2 \sin^{-1} \frac{l}{2r} - \frac{l}{2}\left(r - \frac{a - b}{2}\right)\right]\right\}$$

where

$$r = \frac{l^2 + (a - b)^2}{4(a - b)}.$$

For a very close approximation when a and b are nearly equal, take ACB as an arc of an ellipse of which the bung diameter of the barrel is the minor axis. Then

$$V = 2\pi \int_0^{l/2} y^2 dx = 2\pi B^2 \int_0^{l/2} \left(1 - \frac{x^2}{A^2}\right) dx = 2\pi B^2 \left(\frac{l}{2} - \frac{l^3}{3A^2}\right),$$

where $B^2 = a^2/4$, and hence $A^2 = a^2 l^2 / 4(a^2 - b^2)$. Hence $V = \pi \frac{l}{12}(2a^2 + b^2)$.

MECHANICS.

276. Proposed by V. M. SPUNAR, Chicago, Illinois.

Find the center of gravity of the volume formed by the revolution around the x-axis of the area of the curve $y^4 - axy^2 + x^4 = 0$.

SOLUTION BY RICHARD MORRIS, Rutgers College.

From

$$y^2 = \frac{ax \pm x\sqrt{a^2 - 4x^2}}{2},$$

we find the x limits to be 0 and $a/2$.

Hence

$$\bar{x} = \frac{\int_0^{a/2} \pi(y_1^2 - y_2^2)xdx}{\int_0^{a/2} \pi(y_1^2 - y_2^2)dx}$$

or

$$\bar{x} = \frac{\int_0^{a/2}\left[\frac{ax + x\sqrt{a^2 - 4x^2} - ax + x\sqrt{a^2 - 4x^2}}{2}\right]xdx}{\int_0^{a/2}\left[\frac{ax + x\sqrt{a^2 - 4x^2} - ax + x\sqrt{a^2 - 4x^2}}{2}\right]dx} = \frac{\int_0^{a/2} x^2(a^2 - 4x^2)^{\frac{1}{2}}dx}{\int_0^{a/2} (a^2 - 4x^2)^{\frac{1}{2}}xdx}.$$

Putting $2x = z$, and transforming the numerator only, we get

$$\bar{x} = \frac{\int_0^{a} \frac{1}{8}z^2(a^2 - z^2)^{\frac{1}{2}}dz}{\int_0^{a/2} (a^2 - 4x^2)^{\frac{1}{2}}xdx},$$

which easily integrates by means of tables, giving

$$\bar{x} = \frac{\frac{1}{8}\left[\frac{z}{8}(2z^2 - a^2)\sqrt{a^2 - z^2} + \frac{a^4}{8}\sin^{-1}\frac{z}{a}\right]_0^a}{-\frac{1}{8}\left[(a^2 - 4x^2)^{\frac{3}{2}}\cdot\frac{2}{3}\right]_0^{a/2}} = \frac{3a\pi}{32}.$$

Also solved by S. W. REAVES and H. E. TREFETHEN.

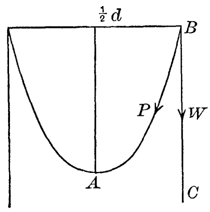
280. Proposed by C. N. SCHMALL, New York City.

Given the distance d between two smooth hooks in the same horizontal line. Show that the shortest string which can form a catenary, with these hooks for points of support, is de where e is the base of the Naperian system of logarithms.

SOLUTION BY A. M. HARDING, University of Arkansas.

Let $y = c/2(e^{\frac{x}{c}} + e^{-\frac{x}{c}})$ be the catenary. Let $BC = l$, then $W = wl$, where w = weight per unit length of the string. The tension at any given point of the catenary is given by $T \sin \theta = ws$, where s = the length of the string measured

from A and θ is the angle which the string makes with the horizontal at the given point. It can be easily shown that $\tan\theta = \sinh x/c$, so that $\sin\theta = \tanh x/c$, and hence $T = ws \coth x/c$.



At the point B we have $s = c \sinh d/2c$ and $T = P$, say. Since the hook is smooth $P = W$ and we have $wl = wc \sinh d/2c \coth d/2c = wc \cosh d/2c$. Hence $l = c \cosh d/2c$ and $\frac{1}{2}L = l + s = c(\cosh d/2c + \sinh d/2c) = ce^{\frac{d}{2c}}$.

We must now find the value of c which makes $\frac{1}{2}L$ a minimum. From the equation $D_c(\frac{1}{2}L) = 0$ we obtain

$$e^{\frac{d}{2c}} - \frac{d}{2c}e^{\frac{d}{2c}} = \tfrac{1}{2}e^{\frac{d}{2c}}\left(2 - \frac{d}{c}\right) = 0, \quad \text{or} \quad c = \frac{d}{2}.$$

Hence $\frac{1}{2}L = d/2 \cdot e = de/2$ or $L = de$.

Also solved by J. B. Smith.

NUMBER THEORY.

187. Proposed by E. T. BELL, New York, N. Y.

If m is any integer, P the product of all the distinct prime factors of m and λ their number, and if $N(x)$ denote the number of divisors of x, then

$$6^\lambda \Sigma N(d)N(m/d) = N(m)N(Pm)N(P^2m),$$

where the sign of summation extends over all the divisors d of m.

Solution by Thomas E. Mason, Indiana University.

Since m has λ distinct prime factors it can be written

$$m = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3} \cdots p_\lambda^{\alpha_\lambda},$$

where the p's are distinct primes. By a well-known formula we have

$$\begin{gathered} N(m) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_\lambda + 1), \quad N(Pm) = (\alpha_1 + 2) \cdots (\alpha_\lambda + 2), \\ N(P^2m) = (\alpha_1 + 3) \cdots (\alpha_\lambda + 3). \end{gathered} \tag{1}$$

Any divisor d of m can be written in the form

$$d = p_1^{r_1}p_2^{r_2}p_3^{r_3} \cdots p_\lambda^{r_\lambda}.$$

Then

$$N(d) = (r_1 + 1)(r_2 + 1) \cdots (r_\lambda + 1),$$

and

$$N(m/d) = (\alpha_1 - r_1 + 1)(\alpha_2 - r_2 + 1) \cdots (\alpha_\lambda - r_\lambda + 1).$$

Now we have

$$
\begin{aligned}
6^\lambda \Sigma N(d)N(m/d) &= 6^\lambda \sum_{r_i=0}^{\alpha_i} (\alpha_1 - r_1 + 1)(\alpha_2 - r_2 + 1)\cdots(\alpha_\lambda - r_\lambda + 1)(r_1+1)(r_2+1) \\
&\qquad\qquad \cdots (r_\lambda + 1) \\
&= 6^\lambda \sum_{r_i=0}^{\alpha_i} (\alpha_1 - r_1 + 1)(r_1 + 1) \cdots (\alpha_\lambda - r_\lambda + 1)(r_\lambda + 1) \\
&= 6 \sum_{r_1=0}^{\alpha_1} [\alpha_1(r_1 + 1) - (r_1^2 - 1)] \cdot 6 \sum_{r_2=0}^{\alpha_2} [\alpha_2(r_2 + 1) - (r_2^2 - 1)] \\
&\qquad \cdots 6 \sum_{r_\lambda=0}^{\alpha_\lambda} [\alpha_\lambda(r_\lambda + 1) - (r_\lambda^2 - 1)].
\end{aligned} \tag{2}
$$

The indicated summations can be readily performed. We have

$$
\sum_{r_i=0}^{\alpha_i} (r_i + 1) = \frac{\alpha_i(\alpha_i + 1)}{2} + \alpha_i + 1 = \frac{(\alpha_i + 1)(\alpha_i + 2)}{2}
$$

and

$$
\sum_{r_i=0}^{\alpha_i} (r_i^2 - 1) = \frac{\alpha_i(\alpha_i + 1)(2\alpha_i + 1)}{6} - \alpha_i - 1 = \frac{(\alpha_i + 1)(\alpha_i + 2)(2\alpha_i - 3)}{6}.
$$

Hence

$$
\begin{aligned}
6 \sum_{r_i=0}^{\alpha_i} [\alpha_i(r_i + 1) - (r_i^2 - 1)] &= 6\left[\alpha_i \frac{(\alpha_i + 1)(\alpha_i + 2)}{2} - \frac{(\alpha_i+1)(\alpha_i+2)(2\alpha_i-3)}{6}\right] \\
&= (\alpha_i + 1)(\alpha_i + 2)(\alpha_i + 3).
\end{aligned}
$$

Substituting for these sums in the last member of (2) and making use of (1), we have

$$
\begin{aligned}
6^\lambda \Sigma N(d)N(m/d) &= (\alpha_1 + 1)(\alpha_1 + 2)(\alpha_1 + 3)(\alpha_2 + 1)(\alpha_2 + 2)(\alpha_2 + 3) \\
&\qquad \cdots (\alpha_\lambda + 1)(\alpha_\lambda + 2)(\alpha_\lambda + 3) \\
&= N(m)N(Pm)N(P^2m).
\end{aligned}
$$

197. Proposed by E. T. BELL, Seattle, Washington.

Show that in the expansion of

$$
\frac{1 + z + z^2 + \cdots + z^{p-1}}{(1 - z)^{p-1}} - 1,
$$

where p is a prime, the coefficients of the various powers of z are divisible by p. [Eisenstein, *Crelle*, t. 27, p. 282.]

SOLUTION BY B. F. YANNEY, University of Wooster.

The given expression is evidently equal to $(1 - z^p)/(1 - z)^p - 1$, which may be put in the form $(1 - z^p)(1 - z)^{-p} - 1$. Expanding the second factor of the first term and noticing that, since p is prime, the coefficient of each term

in the expansion except that of the 1st, the $(p+1)$th, the $(2p+1)$th, and so on to the $(np+1)$th, and so on, is divisible by p, we obtain, omitting the terms with the divisible coefficients,

$$(1-z^p)\left(1+\frac{p(p+1)\cdots(2p-1)}{p!}z^p+\frac{p(p+1)\cdots(3p-1)}{(2p)!}z^{2p}+\cdots\right.$$
$$\left.+\frac{p(p+1)\cdots((n+1)p-1)}{(np)!}\cdots\right)-1.$$

Performing the indicated operations, we get

$$\left(\frac{p(p+1)\cdots(2p-1)}{p!}-1\right)z^p+\left(\frac{p(p+1)\cdots(3p-1)}{(2p)!}-\frac{p(p+1)\cdots(2p-1)}{p!}\right)z^{2p}$$
$$+\cdots+\left(\frac{p(p+1)\cdots((n+1)p-1)}{(np)!}-\frac{p(p+1)\cdots(np-1)}{((n-1)p)!}\right)z^{np}+\cdots.$$

The first coefficient may be written $[(p+1)(p+2)\cdots(p+p-1)-(p-1)!]/(p-1)!$, which is equal to $[pA+(p-1)!-(p-1)!]/(p-1)!$, where A is a polynomial in p. This expression, which is equal to $pA/(p-1)!$, and is an integer, as are all the coefficients, is plainly divisible by p, since the denominator does not contain p as a factor.

The coefficient of the general term may be changed in form to

$$\frac{p(p+1)\cdots(np-1)}{((n-1)p)!}\left(\frac{np(np+1)\cdots(np+p-1)}{[(n-1)p+1][(n-1)p+2]\cdots[np]}-1\right).$$

The first factor of this is an integer; hence so is the second, which latter can be expressed in the form

$$\frac{(np+1)(np+2)\cdots(np+p-1)-[(n-1)p+1][(n-1)p+2]\cdots[(n-1)p+p-1]}{[(n-1)p+1][(n-1)p+2]\cdots[(n-1)p+p-1]},$$

which equals

$$\frac{(npB+(p-1)!)-((n-1)pC+(p-1)!)}{(n-1)pC+(p-1)!},$$

where B and C are polynomials in np and $(n-1)p$, respectively. Since the denominator does not contain p as a factor, but the numerator does, the expression is divisible by p, which completes the proof.

Also solved by ELMER SCHUYLER, H. C. FEEMSTER and the PROPOSER.

MISCELLANEOUS QUESTIONS.

EDITED BY R. D. CARMICHAEL.

Professor F. R. Moulton, of the University of Chicago, calls attention to the fact that the differential equation proposed by Mr. Louis Cohen (see our question 1 in the January issue) belongs to the class given general treatment by him and Dr. W. D. MacMillan in the American Journal of Mathematics, volume 33, pages 63 and 96.

For question 4, proposed in the issue of December, 1913, we need more comprehensive answers than any yet received. We have in hand an excellent reply to question 6; the topic suggested in it is of such interest, however, that several answers are desirable.

QUESTION.

10. What use has been made of regular conference periods for assistance to individual students of secondary and college mathematics, and what services may they render?

REPLY.

5. In what ways may mathematics contribute most to the culture of the individual? What is being done and what may be done to advantage in the matter of developing courses in culture mathematics?

I. REMARKS BY D. N. LEHMER, University of California.

There is, no doubt, a strong prejudice in the minds of many mathematicians against culture courses in mathematics. The phrase is often understood to apply to courses in which there is no attempt to insist on accurate thinking, and where the emphasis is laid not so much on mathematics as upon stories connected with mathematics. The student will learn what an extraordinary amount of time and attention was given by the Alexandrian Greeks to the problem of the trisection of an angle, but he will not know accurately what the problem really meant. As to the reason why their efforts were foredoomed to meet with failure, and just what was the difficulty with which they struggled, these matters will be passed over in haste and perhaps in silence. He will be familiar with all of the great names in mathematics, but he will not have a single clear idea of any of the problems which engaged their attention. These are the "culture courses" that have made the very name an abomination to thoughtful teachers.

The history of mathematics is a great and important subject, but any course based chiefly on that subject will almost certainly be a waste of time for the serious student. The result is not mathematical training at all, but an attempt at historical training and that in a domain where we have not yet developed the rigorous discipline to be found in some subjects. We are all of us familiar with the superficial talker who can converse fluently on the most abstruse subjects without having one clear idea in his head about them. Why should we go to the trouble to plan a course for the development and encouragement of just such creatures?

But there are students whose chief interests lie in other branches of science who would nevertheless like to have an adequate notion of the state of affairs in mathematics. The number of such students is not so small as some people may imagine, and there are enough of them who are ready to go to some trouble in applying themselves to the study to make it worth while to plan a course that shall not be microscopic, nor yet telescopic, and which shall put them in possession of the most important processes and developments of the subject. Such a course has been given at the University of California for a number of years, and while each class has been made the subject of careful experiment, the courses as a whole have been pretty much the same from year to year.

The importance of the notion of one-to-one correspondence in all branches of knowledge would seem to demand that the relation between algebra and geometry be made clear to all serious students. Accordingly it is usual to start the years' work with a drill in the fundamental notions in analytic geometry. The students are freshmen and some may be lacking in trigonometry, but it is possible to teach analytic geometry with a very little trigonometry, and such as is absolutely essential can be supplied in a week or two as is necessary. The circle and conic sections are touched on very lightly, if at all, and progress is made towards the notion of the calculus by means of curves defined by equations of the form $y =$ polynomial in x. Such curves are especially useful in introducing the idea of the derivative, maxima and minima, inflection points, integration and evaluation of areas, volumes, etc. It is believed that it is of considerable importance that the student get the fundamental notions of the calculus with a minimum of effort in the way of actual manipulation of analytic expressions. Even the formal notation used in books on the calculus would seem to be of secondary importance. A student may emerge from such a course never having heard of a reduction formula, but he will know how to state a problem in finding areas or volumes in such a way that it will mean for him the discovery of an anti-derivative.

The course lasts for the whole year, three hours per week. It is possible with some classes to develop certain other ideas such as the foundations of number, or the processes of synthetic projective geometry. The chief emphasis is laid, however, on the principles of the calculus. It is in no sense a "snap" course, and its success is abundantly vouched for in the number of those who come to taste and remain to pursue the subject through the rest of their college career.

This is our notion of a "culture course," and while it is sometimes thought of by the ill-informed as a "hodge-podge" because we do not separate trigonometry and analytic geometry and the calculus into water-tight compartments, it serves the purpose of informing the student of the most important processes of modern mathematics, and at the same time demands from him his utmost efforts.

NOTES AND NEWS.

Florian Cajori, Chairman of Committee.

Dr. Oscar Perron, who since 1910 has been professor of mathematics at Tübingen, is called to a professorship at Heidelberg.

Professor Arnold Emch, of the University of Illinois, has accepted a position on the summer session faculty of the University of California for the coming summer.

Western Reserve University announces the continuation of the teachers' class in mathematics into the second semester. This takes the form of a course in the history of mathematics, conducted by Mr. Alva H. Ford, of the department of mathematics of Adelbert College.

Professor Antonio Favaro, the well-known Italian historian of mathematics and science, has recently made a biographical study of Niccolò Tartaglia. The results of his researches are published in *Isis* for November, 1913, and in a pamphlet, *Per la biografia di Niccolò Tartaglia,* Firenze, 1913.

The Macmillan Company announces for spring publication "The Geometry of Four Dimensions," by Henry Parker Manning, and "Memorabilia Mathematica. The Philomath's Quotation Boook," by Robert E. Moritz, of the University of Washington.

The paper on "Science and mathematics in vocational schools," which was read by Professor Florian Cajori, of Colorado College, before the Central Association of Science and Mathematics Teachers at Des Moines, Iowa, on November 28, has been published in the February number of *School Science and Mathematics*.

The "History of Japanese Mathematics," by D. E. Smith and Yoshio Mikami, announcement of which was made in the Monthly some time since, has appeared from the press of the Open Court Publishing Co. This work places within reach of English readers a field of mathematical history which hitherto has been quite inaccessible.

The thirty-fourth regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, April 10, 11, 1914. There were 74 persons in attendance upon the various sessions, including 47 members of the Society. This was the second meeting of the Section which was designated as a regular meeting of the Society at Chicago. The first was in December, 1913.

The February number of the *School Review* contains an article on "A plan for testing methods of teaching elementary mathematics," by Professor G. W. Myers, of the School of Education, University of Chicago. It contains an account of an interesting experiment made by a high school teacher who undertook to test systematically the heuristic method as against the expository method of teaching first year algebra. He chose for the experiment two of his first year

algebra classes that were, in his judgment, similarly circumstanced and nearly equal in ability. One class was taught by the heuristic method as nearly as the teacher could administer it, and the other was taught entirely by the plan of direct exposition. Mr. Myers makes a plea for more experiments of this kind.

On January 12 occurred the death of CALVIN MILTON WOODWARD, professor emeritus of mathematics and dean of the school of engineering at Washington University, St. Louis, Mo. He was the originator and director of the St. Louis Manual Training School. One of his last important publications is a text on "Applied Mechanics for Engineering and Architectural Students." Professor Woodward was an active and influential member of various scientific and educational organizations, and was once president of the American Association for the Advancement of Science.

Two fellowships in the department of mathematics and two in the department of physics at the Rice Institute will be filled for the academic year 1914–15, each of the annual value of seven hundred and fifty dollars ($750). The successful candidates will be expected to enter upon a course of study and research work leading to the degree of Doctor of Philosophy, and also to assist with elementary teaching in mathematics or physics for about six hours per week. Applications accompanied by testimonials and a full statement of previous work and training should be addressed to the department of mathematics or to the department of physics, The Rice Institute, Houston, Texas.

Mr. CLIFFORD N. MILLS secured the degree of A.M. at the University of Indiana at the close of the last semester and is now acting professor of mathematics at the South Dakota State College at Brookings, in the place of Dr. G. L. Brown, who has been made acting president.

HENRY FREDERICK BAKER, Sc.D., F.R.S., has been elected Lowndean professor of astronomy and geometry at the University of Cambridge in succession to the late Sir Robert Ball. Before this election Dr. Baker was fellow and lecturer of St. John's College, and Cayley university lecturer in mathematics at Cambridge. He has held the presidency of the London Mathematical Society and of the Cambridge Philosophical Society.

In recent years several German professors of mathematics have called public attention to the fact that the number of mathematical students at the various German universities is larger than the probable number of mathematical positions in the German schools. According to the *Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 22 (1913), page 369, the number of mathematical students is much smaller during the current year than it has been during recent years. The number of women students of mathematics is, however, still on the increase in the German universities.

July 24, 1914, is the date for the Napier Tercentenary Celebration, to be held in Edinburgh, Scotland. The celebration will be held under the auspices of the Royal Society of Edinburgh. We quote from a recent circular the following: " The Celebration will be opened on Friday with an Inaugural Address by Lord

of Appeal Sir J. Fletcher Moulton, F.R.S., LL.D., followed by a reception given by the Right Honourable the Lord Provost, Magistrates, and Council of the City of Edinburgh. On Saturday and Monday the historical and present practice of computation and other developments closely connected with Napier's discoveries and inventions will be discussed. . . . Among many who have expressed a warm interest in the Celebration and who hope to take part in the Congress, may be mentioned Professor Andoyer, Paris; Professor J. Bauschinger, Strassburg; Professor Hume Brown, Historiographer Royal for Scotland; Professor Florian Cajori, Colorado, U. S. A.; Professor G. A. Gibson, Glasgow; Dr. J. W. L. Glaisher, Cambridge; Professor Lang, St. Andrews; Professor Macdonald, Aberdeen; Professor E. Pascal, Naples; Professor Karl Pearson, London; Professor David Eugene Smith, New York; Professor Steggall, Dundee; Professor Whittaker, Edinburgh. . . . Relics of Napier, collected by Lord Napier and Ettrick and other representatives of the family, will also be on view; and it is intended to bring together for exhibition books of Tables and forms of Calculating Machines, which may reasonably be regarded as natural developments of the great advance made by Napier." It is planned to issue a memorial volume containing the addresses and papers read before the Congress, and other material of historic and scientific value. Mathematicians from all countries are invited to attend the celebration.

It will be recalled that the floods last year prevented the Ohio teachers of mathematics and science from holding their annual meeting. Hence the following report from Professor S. E. Rasor, Vice-President of the association, is most gratifying.

"The tenth annual meeting of the Association of Ohio Teachers of Mathematics and Science was held at the Ohio State University, Columbus, Ohio, Friday evening, April 3 and Saturday, April 4. The Friday evening address was given by James F. Barker of the Cleveland East Technical High School. On Saturday the mathematics section listened with much interest and lasting impression to the addresses by Professor H. E. Slaught of the University of Chicago. Papers and discussions were also presented before this section by a number of prominent Ohio teachers. A large and most enthusiastic and representative body of teachers from the colleges and secondary schools of the state were present and declared the meeting to be the best in the history of the Association. A significant feature of the meeting was the fact that of those present about ninety-five per cent. were men. The members of the association were guests of the University at a complimentary noon-day luncheon on Saturday. Officers for the coming year were elected as follows: President, C. C. Morris, Ohio State University, Columbus, Ohio; Vice-President, S. J. Mauckley, Woodward High School, Cincinnati, Ohio; Secretary-Treasurer, E. W. E. Schear, Ottebein University, Westerville, Ohio; Assistant Secretary, Miss Margaret Devereaux, Alliance High School, Alliance, Ohio."

The twenty-sixth educational conference of the academies and high schools in relation with the University of Chicago was held at the University on Friday

and Saturday, April 17–18, 1914. At the mathematics section two leading questions were discussed (1) The comparative merits of home study and supervised study at the school; (2) Reviews of mathematical literature for teachers.

The first topic was presented by Mr. C. M. Austin, of the Oak Park High School, who gave a resumé of an extended paper on this subject by Mr. E. R. Breslich, based upon a careful investigation at the high school connected with the University of Chicago School of Education.

The second topic was introduced by Mr. H. C. Wright, of the University High School and Professor H. E. Slaught, of the University of Chicago. Mr. Wright presented a resumé of the chief articles published in the *Mathematics Teacher* during the past six years, and Professor Slaught explained the character of the literature published in the MONTHLY since it was reorganized in January, 1913, and dedicated to the interests of teachers of mathematics in the early collegiate and advanced secondary fields. The fact was brought out and emphasized that in our associations of secondary teachers of mathematics the discussions during the past ten or twelve years have dealt primarily with pedagogical and administrative schemes and that the journals representing these associations, such as the *Mathematics Teacher* and *School Science and Mathematics,* have properly and naturally reflected these aspects. On the other hand, little or no attention has been given to the *teachers themselves* who are by far the most important factor in any educational scheme, and who need sources of inspiration and power outside of their daily routine. Such inspiration and quickening of enthusiasm the AMERICAN MATHEMATICAL MONTHLY is trying to provide and it is the only journal in this country occupying this field. Special reference was made to the study of synthetic projective geometry as absolutely essential to provide a background and a viewpoint for the teacher of elementary plane geometry. It is believed that a text-book will be soon forthcoming which will be suitable not only for use in first or second year college classes, by those who are preparing to teach, but also for private study in small groups by teachers now in the service.

Errata. The following errors have been noted in the December issue, 1913. On page 304, line 10 up should read $z^3 = -6r\rho^2$; line 8 up should read $\rho = \pm 6k\sigma^3$; line 5 up should read $y = p^3 \mp 6\sigma^3$. On page 305, line 19 up, the first double sign should be $\mp$.

All copies of the MONTHLY for January, 1913, have been exhausted. The demand for sample copies was so great for this particular number that the supply was entirely inadequate. Any one who may know of extra copies not belonging to sets will confer a great favor by informing the MANAGING EDITOR.

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ON SOME GEOMETRIC PROPERTIES OF CIRCULAR TRANSFORMATIONS.

By ARNOLD EMCH, University of Illinois.

§ 1. **Introduction.** In what follows I shall give simple demonstrations of a few geometric theorems that are of value in the study of ring-shaped domains as used in the definition of certain special automorphic functions.[1] As will be seen later on, the theory of such domains is closely connected with linear substitutions or circular transformations in a complex plane, as defined in § 3. Although there is nothing essentially new as to content, it will be at once apparent, how simply, in comparison with ordinary analytic methods, some of the theorems may be proved by making use of a fundamental proposition in group-theory. With this fact I want to emphasize the importance of an early introduction of linear substitutions and their principal group-properties in certain mathematical courses, as for example in function-theory.

A *ring-shaped domain* G may be defined as a connected portion of a complex plane bounded by two non-intersecting circles (see Fig. p. 140). Among such domains are included those into which G passes when the two circles become tangent.

By a *group* we understand a class of operations, such that the product of any two operations of the class is again an operation of the same class. The interpretation of this definition for linear substitutions will be found in § 3.

§ 2. **Reflexion on a Circle.** By *reflexion* of a point P on *a straight line* l we define a point P' on a line through P perpendicular to l, such that the distances of the distinct points P and P' from l are the same. The transformation by reciprocal radii with respect to a fixed circle is called *reflexion on the circle.*

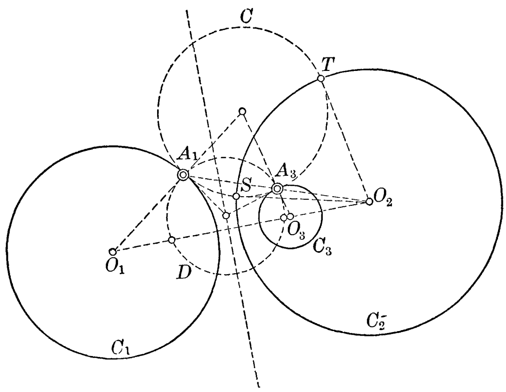
An *inversion* with respect to a fixed circle may be defined as a reflexion on that circle followed by a reflexion on a fixed axis through the center of the same circle. This circle may, of course, be located anywhere in the plane.

[1] SCHOTTKY: "Ueber die Funktionenklasse die der Gleichung $F(\alpha x + \beta/\gamma x + \delta) = F(x)$ genügt," *Crelle's Journal,* Vol. 143, pp. 1–24 (June, 1913). See also the same *Journal,* Vol. 101, pp. 231–236 (1887).

Two circles C_1 and C_3 determine a *pencil of circles*, by which we understand the totality of circles passing through the points of intersection of C_1 and C_3. The pencil is called *hyperbolic* when the points of intersection are real and distinct; *parabolic*, when they are coincident; *elliptic*, when they are imaginary. Denoting the radii of C_1 and C_3 by r_1 and r_3, and the distance between their centers by e, the distance ϵ between the null-circles of the pencil is easily found to be

$$\epsilon = \frac{2r_1r_3}{e}\sqrt{\frac{(r_1^2 + r_3^2 - e^2)^2}{4r_1^2r_3^2} - 1}. \tag{1}$$

This expression is real, vanishes, or is imaginary, according as C_1 and C_3 do not



intersect (in real points), are tangent, or do intersect. The angle θ at which the two circles intersect is defined by

$$\sin^2\theta = \frac{(r_1^2 + r_3^2 - e^2)^2}{4r_1^2r_3^2} - 1 \tag{2}$$

and is imaginary in the elliptic, vanishes in the parabolic, and is real in the hyperbolic case. This expression for $\sin^2\theta$ is important in a certain proof of convergence given by Schottky, *loc. cit.*

A circle C_2 may now be constructed so that C_1 and C_3 are reflexions of each other on C_2. Every circle C tangent to both C_1 and C_3 at points A_1 and A_3, which are reflexions on C_2, cuts this circle orthogonally. See Fig. By any other reflexion on a circle L, the circles C_1, C_2, C_3, C are transformed into circles C_1', C_2', C_3', C', so that all circles C' are orthogonal to C_2'. Consequently C_1' and C_3' are reflexions on C_2'. We may state this as

THEOREM 1. *By a reflexion on a circle L three circles C_1, C_2, C_3 of which C_1 and C_3 are reflexions on C_2 are transformed in the same order into three circles C_1', C_2', C_3' of which C_1' and C_3' are reflexions on C_2'.*[1]

When the center of L is on C_2, then C_2 is transformed into a straight line C_2', and C_1' and C_3' become equal circles and are ordinary reflexions on C_2'.

[1] This, in another form, corresponds to theorem 3, p. 241, of *Osgood's Lehrbuch der Funktionentheorie*, Vol. 1, 2d ed. (1912).

An elliptic pencil with the origin and a point on the real axis at a distance e as limiting points may be represented by

$$(x - e)^2 + y^2 - \lambda(x^2 + y^2) = 0$$

or

$$\left(x - \frac{e}{1-\lambda}\right)^2 + y^2 = \frac{\lambda e^2}{(1-\lambda)^2}, \tag{3}$$

for all real positive values of λ. For a definite λ, the distances of the limiting points from the center of the circle (3) are

$$\frac{e}{1-\lambda} \quad \text{and} \quad \frac{e}{1-\lambda} - e = \frac{\lambda e}{1-\lambda}.$$

The product of the two distances, $\lambda e^2/(1-\lambda)^2$, equals the square of the radius of the circle (3). From this it follows immediately that *the limiting points of an elliptic pencil are reflexions on every circle of the pencil* and we have

THEOREM 2. *A reflexion on any circle of an elliptic pencil transforms the pencil into itself. The same is, of course, also true of an hyperbolic and a parabolic pencil, since all circles of a pencil pass through two fixed points, which are imaginary in an elliptic, real and distinct in an hyperbolic and coincident in a parabolic pencil.*

In the circle (3) the distances of the extremities of the diameter on the axis from the origin, one of the limiting points, are

$$\frac{e}{1-\lambda} + \frac{e\sqrt{\lambda}}{1-\lambda} = \frac{e}{1-\lambda}(1 + \sqrt{\lambda}),$$

and

$$\frac{e}{1-\lambda} - \frac{e\sqrt{\lambda}}{1-\lambda} = \frac{e}{1-\lambda}(1 - \sqrt{\lambda}).$$

Inverting this we get for the center of the reflected circle (3) on the unit-circle around the origin the distance

$$\frac{1}{2}\left\{\frac{1-\lambda}{e(1+\sqrt{\lambda})} + \frac{1-\lambda}{e(1-\sqrt{\lambda})}\right\} = \frac{1}{e},$$

an expression which is independent of λ. Hence

THEOREM 3. *The reflexion of an elliptic pencil on a circle having one of the limiting points as a center is a concentric pencil with the reflexion of the other limiting point as a center.*[1]

§ 3. **Circular Transformations.**[2] **Groups of Substitutions.** A linear transformation of a complex variable

[1] A great number of other theorems on reflexion may be found in J. CASEY: *A treatise on Analytic Geometry*, chapt. III (1893); R. STURM: *Die Lehre von den geometrischen Verwandtschaften*, Vol. IV, pp. 72–95 (1909); W. FIEDLER: *Cyklographie* (1882).

[2] A comprehensive study of such transformations may be found in an article by F. N. Cole on "The linear functions of a complex variable," *Annals of Mathematics*, Vol. V, pp. 121–176 (1890).

$$z' = \frac{az + b}{cz + d}, \tag{4}$$

has the well-known property that it transforms circles into circles (including straight lines) and is therefore frequently called a circular transformation. If we transform the point z' by another circular transformation

$$z'' = \frac{a_1 z' + b_1}{c_1 z' + d_1}, \tag{5}$$

then the relation between z'' and z is given by

$$z'' = \frac{(aa_1 + cb_1)z + (ba_1 + db_1)}{(ac_1 + cd_1)z + (bc_1 + dd_1)}, \tag{6}$$

that is, z'' results from z by another circular transformation. Denoting the first substitution by which z is replaced by z' symbolically by S, similarly the second, by which z' passes into z'', by T and the third by U, we may say that the product of the substitutions S and T is U, or that S followed by T gives U. This may be symbolically written:

$$ST = U.$$

If some other substitution R carries the point z'' into z''', then there is again a linear substitution (geometrically a circular transformation) which changes z directly into z'''. This substitution may be symbolically written in the form

$$STR = U.$$

In this combination of substitutions the associative law holds, that is,

$$S(TR) = (ST)R,$$

while, in general, the commutative law is not true; namely

$$ST \neq TS.$$

If S is given, then by the inverse S^{-1} of S we understand the substitution of

$$\frac{dz - b}{-cz + a}$$

for z, which is obtained by solving

$$z = \frac{az' + b}{cz' + d}$$

or z'. It is easily seen that SS^{-1} is a substitution which leaves z invariant and is therefore called the identical substitution and may be indicated by

$$SS^{-1} = 1.$$

The foregoing definitions are sufficient to convey the idea of the group. For the sake of brevity this discussion is confined to finite groups.

A finite group of substitutions consists of a finite number of substitutions such that the product of any two of these is again one of the substitutions. The inverse of any substitution of the group belongs to the group. The group contains the identical substitution.

Such a group is obtained, for example, if we take n substitutions each S with the property that

$$SSS \cdots S \equiv S^n = 1.$$

Now $1, S, S^2, S^3, \cdots, S^{n-1}$ form a finite group of order n, which is at the same time cyclic. The reason for the attribute "cyclic" is apparent from the arrangement of the substitutions of the group.

§ 4. **Involutory Transformations.** An involutory transformation, or simply an involution in a complex plane may be defined as

$$z' = \frac{az + b}{cz - a}. \tag{7}$$

From this we find

$$z = \frac{az' + b}{cz' - a}.$$

If we transform by (7) z into z' and by the same transformation

$$z'' = \frac{az' + b}{cz' - a},$$

z' into z'', we find $z'' = z$, that is, by repeating an involution twice in succession, the original point z is transformed back into itself. Denoting the involutory substitution (7) by T, we have therefore, according to § 3,

$$T^2 = 1, \tag{8}$$

so that $1, T$ are the substitutions of a finite group of order 2, which is of course cyclic.

We shall now prove

THEOREM 4. *An involution in a complex plane is an inversion with respect to a fixed circle and a fixed axis through its center.*[1]

For this purpose turn the point z and its involutoric $(az + b)/(cz - a)$ through an angle ϕ about the origin. To their new positions apply a translation α, so that after the combined motion the points will be in the positions

$$e^{i\phi}z + \alpha \quad \text{and} \quad e^{i\phi}\frac{az + b}{cz - a} + \alpha.$$

[1] In a different form this proposition is also stated and proved in HOLZMÜLLER: *Einführung in die Theorie der isogonalen Verwandtschaften*, pp. 43–46 (1882).

If it is possible to determine ϕ, α and a real positive quantity ρ such that

$$(9) \qquad (e^{i\phi}z + \alpha)\left(e^{i\phi}\frac{az+b}{cz-a} + \alpha\right) = \rho^2,$$

for all values of z, the proposition will be proved.

Equation (9) may be written in the form

$$e^{i\phi}(e^{i\phi}a + c\alpha) + \{e^{i\phi}(e^{i\phi}b - a\alpha) + \alpha(e^{i\phi} + c\alpha) - c\rho^2\}z + \alpha(e^{i\phi} - a\alpha) + a\rho^2 = 0.$$

We have therefore the conditions

$$\begin{aligned} e^{i\phi}a + c\alpha &= 0, \\ e^{i\phi}(e^{i\phi}b - a\alpha) &= c\rho^2, \\ \alpha(e^{i\phi}b - a\alpha) &= -a\rho^2. \end{aligned}$$

But the first of these is a consequence of the last two. From these is found

$$(10) \qquad \alpha = \pm\frac{a\rho}{\sqrt{a^2+bc}}, \qquad e^{i\phi} = \mp\frac{c\rho}{\sqrt{a^2+bc}}.$$

As the absolute value of $e^{i\phi}$ is 1, we find for ρ

$$(11) \qquad \rho = \left|\frac{\sqrt{a^2+bc}}{c}\right|,$$

which is easily recognized as half the distance between the double-points of the involution. From (9) and (10) it would appear that the problem has two solutions. To show that the inversion is unique take first in (9) and (10) for α and $e^{i\phi}$ the positive and negative sign and designate the corresponding values by α_1 and ϕ_1. Taking $\alpha = \alpha_1$ and $\phi = \phi_1$, then

$$e^{i\phi_1}\cdot z \quad \text{and} \quad e^{i\phi_1}\cdot\frac{az+b}{cz-a}$$

are inverse with respect to a circle having $-\alpha_1$ as a center, ρ as given by (11) as a radius, and the line l through $-\alpha_1$ parallel to the real axis as an axis. Rotating the figure of this inversion back through an angle $-\phi_1$ we obtain the original points z and $(az+b)/(cz-a)$ and $-\alpha_1$ is moved to a position $\beta = -\alpha_1 e^{i\phi}$. The axis l is moved to a position g through β. But $\beta = +(a/c)$ is the midpoint between the double-points. *Consequently* z *and* $(az+b)/(cz-a)$ *are inverse with respect to a circle through the double-points of the transformation as extremities of a diameter and with the line joining them as an axis.*

Taking for α the other sign $\alpha_2 = -\alpha_1$, and $\phi_2 = \phi_1 + \pi$, we get the same result.

§ 5. **General Circular Transformation of an Inversion and an Ordinary Reflexion.** It is well-known that the transformed of a group is an isomorphic

group, that is, a group which obeys the same formal laws as the original group. Hence if we transform the cyclic group of order 2

$$\left(z,\ \frac{az+b}{cz-a}\right)$$

by any circular transformation

$$z' = \frac{\alpha z+\beta}{\gamma z+\delta},$$

and designate the substitutions $(az + b)/(cz - a)$ and $(\alpha z + \beta)/\gamma z + \delta)$ by the symbols S and T respectively, the transformed group

$$(1,\ T^{-1}ST)$$

is also of order 2, and is consequently geometrically represented by an involution, or an inversion on a circle.

This result may be stated as

THEOREM 5. *A circular transformation of an inversion is again an inversion.* In connection with theorem (1) we deduce at once

THEOREM 6. *A circular transformation of a reflexion on a circle is again a reflexion on a circle, or on a straight line.*[1]

Consider now the group of order 2,

$$(z,\ 2\lambda - z),$$

where λ is a constant, and which geometrically is represented by an ordinary reflexion on a straight line parallel to the axis of imaginaries at a distance λ from the origin. The transformed of this group by any circular transformation is again of order 2 and is geometrically represented by a reflexion on the circle which is the transformed of the line $x = \lambda$ in the z-plane.

Thus we can state

THEOREM 7. *A circular transformation transforms a system of parallel equidistant straight lines into a parabolic pencil of circles, so that of any three consecutive circles C_1, C_2, C_3, we may consider C_1 and C_3 as inversions or reflexions on C_2.*

§ 6. **Geometric Properties of Ring-Shaped Domains Connected with Infinite Cyclic Groups of Linear Substitutions.** As is well known every linear substitution between two complex variables may be written in one of the two forms,

$$(12)\qquad \frac{x_1-a}{x_1-b} = q\frac{x-a}{x-b}, \qquad (13)\qquad \frac{A}{x_1-a} = \frac{A}{x-a}+1,$$

where x_1 and x are the variables and a and b are the double-points. In (12) we have the loxodromic case when $|q| \lesseqgtr 1$, elliptic when $|q| = 1$. In (13) we have the parabolic substitution. I shall confine myself to the loxodromic and parabolic

[1] Corresponds to theorem 4, p. 244, Osgood, *loc. cit.*

cases. Repeating the substitutions (12) and (13) each λ times in succession we get

$$\frac{x_\lambda - a}{x_\lambda - b} = q^\lambda \frac{x - a}{x - b}, \tag{14}$$

$$\frac{A}{x_\lambda - a} = \frac{A}{x - a} + \lambda, \tag{15}$$

representing, when λ assumes all integral values between $-\infty$ and $+\infty$, cyclic groups. Consider first the loxodromic group, where $|q| \neq 1$. By the linear substitution

$$c \cdot \frac{x - a}{x - b} = z, \tag{16}$$

the x-plane is transformed into the z-plane, and the substitutions of the transformed group assume the simple form

$$z_\lambda = q^\lambda z. \tag{17}$$

As the fundamental domain of this group we choose the ring-shaped surface between the unit-circle and the concentric circle of radius $|q|$. The remaining domains are bounded by the pencil of concentric circles with radii $|q^\lambda|$, where $-\infty < \lambda < +\infty$. Since for any three consecutive circles C_λ, $C_{\lambda+1}$, $C_{\lambda+2}$ there is

$$|q^\lambda| \cdot |q^{\lambda+2}| = |q^{\lambda+1}|^2,$$

we conclude that C_λ and $C_{\lambda+2}$ are reflexions on $C_{\lambda+1}$. Transforming back into the x-plane the pencil (C_λ) is transformed into the elliptic pencil with a and b as limiting points. The transformed circles K_λ corresponding to those of C_λ form the boundaries of the domains belonging to the group (14). From theorem (6) we conclude immediately:

THEOREM 8. *Of any three consecutive circles K_λ, $K_{\lambda+1}$, $K_{\lambda+2}$ of the domains associated with the cyclic loxodromic group, K_λ and $K_{\lambda+2}$ are reflexions on $K_{\lambda+1}$.*

In case of the parabolic group we may transform it by the substitution

$$\frac{A}{x - a} + c = z \tag{18}$$

into the z-plane, so that we get

$$z_\lambda = z + \lambda. \tag{19}$$

As the fundamental region we choose the strip between the axis of imaginaries and a line parallel to it at a distance equal to unity. The domains are now bounded by a system of parallel equidistant lines. Hence according to theorem (7) in the corresponding pencil of the x-plane of any three consecutive circles K_λ, $K_{\lambda+1}$, $K_{\lambda+2}$ we find as in the foregoing theorem that K_λ and $K_{\lambda+2}$ are reflexions on $K_{\lambda+1}$. These properties which are well-known results[1] also follow from the theory of conformal transformations.

[1] See § 16, p. 200, Vol. 1, *Modulfunctionen*, by Klein-Fricke.

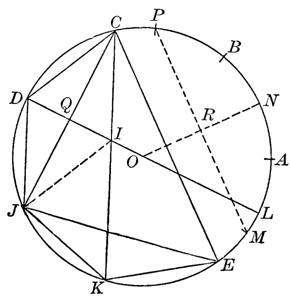
CONCERNING THE REGULAR INSCRIBED HEPTAGON.

By S. A. JOFFE, New York City.

Mr. J. Q. McNatt's article " A Geometrical Discussion of the Regular Inscribed Heptagon," which appeared in the January number of this MONTHLY, pages 13 and 14, contains a very interesting and ingenious method of arriving at what the author calls " The Heptagon Cubic," from the solution of which he shows that the square of the side of the regular heptagon equals $\frac{3}{4}$, as a very close approximation.

Without detracting from the value of the method, I should like to point out that some of the details in the process may be omitted or simplified, and it has occurred to me that it may be worth while to present here the method in an abbreviated form, thus making the matter more widely known. In preparing the following lines, I have endeavored to preserve as much of the original figure and of the author's text as was consistent with the object of simplification.

To calculate the length of the side of a regular inscribed heptagon in terms of the radius as a unit, suppose, in the accompanying figure, that the circumference is divided into seven equal parts at the points A, B, C, D, J, K, and E.



Draw the diameter DL and the chords JE, CJ, CK and CE, and let CJ and CK cut DL at Q and I respectively. Join I and J.

Let the length of the side of the regular inscribed heptagon be h units. We have $DL : DC :: DC : DQ$; hence $DQ = \frac{1}{2}h^2$. Moreover, $QC = \sqrt{DC^2 - DQ^2} = \sqrt{h^2 - \frac{1}{4}h^4} = \frac{1}{2}h\sqrt{4 - h^2}$. From the isosceles triangle DCI we have $CI = CD = h$, and $DI = 2DQ = h^2$; from the isosceles triangle CIJ we have $IJ = CI = h$, and $CJ = 2CQ = h\sqrt{4 - h^2}$, so that

$$JE = h\sqrt{4 - h^2}. \tag{1}$$

Now, by geometry $IL \times DI = CI \times IK$, and since $IL = DL - DI = 2 - h^2$, we have $(2 - h^2) \times h^2 = h \times IK$; hence

$$IK = 2h - h^3, \tag{2}$$

and since $CK = CI + IK = 3h - h^3$, and $CE = CK$, therefore

$$CE = 3h - h^3. \tag{3}$$

The two isosceles triangles CJE and IJK, having equal angles at E and K respectively, are similar. Hence we have $JE : CE :: JK : IK$, or, using (1), (2) and (3)

$$h\sqrt{4-h^2} : (3h - h^3) :: h : (2h - h^3).$$

Hence

$$(4') \qquad (2 - h^2)\sqrt{4 - h^2} = 3 - h^2.$$

Squaring, we get

$$(4 - 4h^2 + h^4)(4 - h^2) = 9 - 6h^2 + h^4,$$

and expanding,

$$16 - 20h^2 + 8h^4 - h^6 = 9 - 6h^2 + h^4.$$

Finally, transposing and simplifying, we obtain the author's *Heptagon Cubic:*

$$(5) \qquad 7 - 14h^2 + 7h^4 - h^6 = 0.$$

Solving by Horner's method, we find $h^2 = .7530203962821\ldots = \frac{3}{4}$, approximately.

Remark. It will thus be seen that while there is introduced a new line IJ, we dispense with the consideration of the line OC, and with both the consideration and the computation of the author's lines SE, SJ, SK and SC. As a result, the equation (4′) appears in a much simpler form than the author's equation (4).

The approximate construction of the heptagon may also be simplified as follows:

Let M, N and P be three consecutive vertices of an inscribed regular hexagon. Draw the chord MP and the radius ON, and let MP meet ON in R. Then MR is, approximately, the length h of the side of the regular inscribed heptagon. The reason is self-evident: approximately, $h = \frac{1}{2}\sqrt{3}$, and MP, as the side of a regular inscribed triangle, $= \sqrt{3}$, so that $MR = \frac{1}{2}\sqrt{3}$, and therefore $MR = h$, approximately.

A PROBLEM IN NUMBER THEORY.

By GEO. A. OSBORNE, Massachusetts Institute of Technology.

§ 1. When is the sum of the squares of two successive integers a perfect square? The following are examples:

$$3^2 + 4^2 = 5^2, \quad 20^2 + 21^2 = 29^2. \text{ The next is } 119^2 + 120^2 = 169^2.$$

The numbers 3, 20, 119, . . . are the terms of a series

$$0,\ 3,\ 20,\ 119,\ 696,\ \cdots\ u_n,\ u_{n+1}, \tag{1}$$

where

$$u_{n+1} = 6u_n - u_{n-1} + 2. \tag{2}$$

This may be proved as follows:

From (2), which is the relation between any three successive terms of (1), we may derive the relation between any two successive terms as follows:

From (2)

$$u_{n+1} + u_{n-1} = 6u_n + 2,$$

$$u_{n+1}^2 - u_{n-1}^2 = (6u_n + 2)(u_{n+1} - u_{n-1}),$$

$$(u_{n+1} - 1)^2 - 6u_n u_{n+1} = (u_{n-1} - 1)^2 - 6u_n u_{n-1},$$

Adding to each member $(u_n - 1)^2$, we have

$$(u_{n+1} - 1)^2 + (u_n - 1)^2 - 6u_n\, u_{n+1} = (u_n - 1)^2 + (u_{n-1} - 1)^2 - 6u_{n-1}u_n, \tag{3}$$

which is of the form $f(n) = f(n - 1)$.

Hence by induction, $f(n) = c$, a constant independent of n. By applying the first member of (3) to the terms 3 and 20, we find $c = 5$. Hence

$$(u_{n+1} - 1)^2 + (u_n - 1)^2 - 6u_n u_{n+1} = 5 \tag{4}$$

is the relation between any two successive terms of (1).

Solving (4) with respect to u_{n+1}, we have

$$u_{n+1} = 3u_n + 1 \pm 2\sqrt{2u_n^2 + 2u_n + 1}, \tag{5}$$

in which the lower sign is rejected since, otherwise, the right member would be less than u_n. It follows from (5) that

$$2u_n^2 + 2u_n + 1 = \text{a square},$$

that is,

$$u_n^2 + (u_n + 1)^2 = \text{a square},$$

one part of the result which was to be proved.

§ 2. From the terms of (1) we may write

$$\begin{aligned}
0^2 + 1^2 &= 1^2,\\
3^2 + 4^2 &= 5^2,\\
20^2 + 21^2 &= 29^2,\\
119^2 + 120^2 &= 169^2,\\
696^2 + 697^2 &= 985^2,\\
4059^2 + 4060^2 &= 5741^2,\\
23660^2 + 23661^2 &= 33461^2,\\
137903^2 + 137904^2 &= 195025^2,\\
803760^2 + 803761^2 &= 1136689^2,\\
\cdots\quad\cdots &\quad\cdots\\
\cdots\quad\cdots &\quad\cdots
\end{aligned}$$

The second members are terms of a series,

$$1,\ 5,\ 29,\ \cdots\ u_n,\ u_{n+1},$$

where

$$u_{n+1} = 6u_n - u_{n-1}.$$

§ 3. It remains to be shown that the terms of the series (1) are the *only* integers that satisfy the condition

$$N^2 + (N + 1)^2 = \text{a square.} \tag{6}$$

If from (4) we express u_n in terms of u_{n+1}, we have

$$u_n = 3u_{n+1} + 1 - 2\sqrt{2u_{n+1}^2 + 2u_{n+1} + 1}, \tag{7}$$

from which

$$u_{n-1} = 3u_n + 1 - 2\sqrt{2u_n^2 + 2u_n + 1}. \tag{8}$$

Consider the equation

$$y = 3x + 1 - 2\sqrt{2x^2 + 2x + 1}. \tag{9}$$

Then

$$2y^2 + 2y + 1 = 2(3x + 1 - 2\sqrt{2x^2 + 2x + 1})^2 + 2(3x + 1 - 2\sqrt{2x^2 + 2x + 1}) + 1$$

$$= (4x + 2 - 3\sqrt{2x^2 + 2x + 1})^2.$$

Hence if

$$2x^2 + 2x + 1 = \text{a square,}$$

then also

$$2y^2 + 2y + 1 = \text{a square.}$$

That is, if x satisfies (6), so does y. Comparing (9) with (7) and (8), it appears that if $x = u_{n+1}$, $y = u_n$; and if $x = u_n$, $y = u_{n-1}$. And as y is an increasing function of x, since

$$\frac{dy}{dx} = 3 - \frac{4x + 2}{\sqrt{2x^2 + 2x + 1}} > 0,$$

it follows that if $u_n < x < u_{n+1}$, then $u_{n-1} < y < u_n$. That is, if there is an integer satisfying (6) between u_n and u_{n+1}, there is another such integer between u_{n-1} and u_n.

There is no integer satisfying (6) between the terms 3 and 20; hence there is none between 20 and 119, and consequently none between any two successive terms of the series (1).

§ 4. In the list of equations in § 2, it may be noticed that none of the successive integers end in 2, 5 or 8. As the final digits recur, it follows that

The sum of the squares of two successive integers, one of which ends in 2, 5 *or* 8, *cannot be a perfect square.*

NOTE ON PERFECT MAGIC SQUARES FOR 1914.

By B. L. REMICK, Manhattan, Kansas.

In the February MONTHLY, Mr. V. M. Spunar states that the only values of n in Zerr's formula yielding positive elements for the 1914 magic square are 3 and 11. These are indeed the only "prime" values of n. He has, however, overlooked $n = 4$ and $n = 12$, which yield the following squares with elements all positive:

$n = 4$

471	485	484	474
482	476	477	479
478	480	481	475
483	473	472	486

$n = 12$

231	89	229	91	227	93	94	224	96	222	98	220
100	218	102	216	104	214	213	107	211	109	209	111
207	113	205	115	203	117	118	200	120	198	122	196
124	194	126	192	128	190	189	131	187	133	185	135
183	137	181	139	179	141	142	176	144	174	146	172
148	170	150	168	152	166	165	155	163	157	161	159
160	158	162	156	164	154	153	167	151	169	149	171
147	173	145	175	143	177	178	140	180	138	182	136
184	134	186	132	188	130	129	191	127	193	125	195
123	197	121	199	119	201	202	116	204	114	206	112
208	110	210	108	212	106	105	215	103	217	101	219
99	221	97	223	95	225	226	92	228	90	230	88

BOOK REVIEWS.

A History of Japanese Mathematics. By DAVID EUGENE SMITH and YOSHIO MIKAMI. The Open Court Publishing Company, Chicago, 1914. vii + 288 pages. $3.50.

This book is a beautiful specimen of the printer's art. The paper, the type, and the illustrations make it a work which it is a delight to read almost aside from the text. I hasten to add that text and content are in harmony with the dress. The sympathetic portrayal of the development of Japanese mathematics, largely indigenous and as the authors well state, "like her art, exquisite rather than grand," will appeal to a wide circle of readers and will contribute to a juster and broader appreciation of the Japanese genius.

The history of Japanese mathematics is divided into six periods. The first extends to 552 A. D. and was almost entirely a native development. The second period, from 552 to 1600, was characterized by the predominance of Chinese mathematics. The third period was a kind of Renaissance culminating in the appearance of Seki, the most famous Japanese mathematician. The fourth period, 1675 to 1775, and the fifth, 1775 to 1868, are marked by the complete development of the native mathematics, the *wasan*, in the second of these two periods somewhat influenced by European mathematics. The sixth is the period of the present, the mathematics of the world which knows nothing of political and racial boundaries.

The earliest and second periods are treated in 17 pages which is significant of the fact that little is known definitely of the history of these times. The development of the *soroban* from the Chinese *swan-p'an* and the methods of operating with this instrument are given in a detailed and satisfactory manner. Incidentally it seems worthy of note that *swan-p'an* with the meaning in Chinese of "reckoning table" corresponds to the Greek word from which we have "abacus," also having the meaning "table," particularly for bankers. The "sangi" or numeral rods, another type of palpable arithmetic, are explained both as used for representing numbers and also as applied to algebra. Numerical approximation of the roots of equations, especially in connection with the circle, made a particular appeal to the Japanese as well as to the Chinese. Magic squares and even magic circles received much attention in the third period. Three chapters are given over to Seki Kōwa, 1642–1708, and his pupils, including an extended discussion of the "yenri" or circle principle, an approach to the methods of the calculus. The remaining six chapters are devoted respectively to the eighteenth century, Ajima Chokuyen, the opening of the nineteenth century, Wada Nei, the close of the old Wasan, and the introduction of occidental mathematics. An index and vocabulary add to the usefulness of the work.

The typographical errors and other slips which I have noted are few in number and unimportant: p. 15 "compotus rolls," probably better "computus rolls"; p. 19, n. 6, "I" for "we"; p. 29, "aboud"; p. 31, "nstrument"; p. 51, "Latin *res* and the Italian *cosa*, both of which had undoubtedly come from the

East," but we know that these are direct translations from the Arabic *shai*, meaning *thing*; p. 51, n. 4, Egyptian *hau* should be *ahau* for *heap;* p. 80, "Arabic numerals," elsewhere "Hindu-Arabic numerals"; p. 279, "develpoed."

In every way this work can be commended to the student of the history of science and to the student of Japanese civilization, and quite as much to the general reader, for the greater part of the story is not at all technical. May this beautiful product of a German printer, W. Drugulin, Leipzig, put out by an American publishing house, the joint work of a Japanese and an American, be symbolical of a better mutual understanding between these countries.

L. C. KARPINSKI.

The Development of Mathematics in China and Japan; Abhandlungen zur Geschichte der mathematischen Wissenschaften, Vol. XXX. By YOSHIO MIKAMI. Teubner, Leipzig, 1913. G. E. Stechert and Co., New York. x + 347 pages. $5.50 net.

In 1910 Mr. Mikami published in the same series as this volume the work, *Mathematical Papers from the Far East,* which attracted favorable attention. The present work is of great importance because it brings systematized information about the history of mathematics in China and Japan, based upon a study of such original documents as remain. Regret must be expressed that the book is marred, even as the preceding work, by faulty English. The prefatory note by G. B. Halsted implies that the task of correcting the English was entrusted to him, but instead of correcting he says: "This is not the idiom of England nor of the United States nor have I striven so to cramp it." I take it that Mr. Mikami desired to bring out the work in idiomatic English and censure for the many errors of construction as well as the actually unintelligible statements can be attached only to the American scholar to whom the revision was entrusted. I think it is also fair to say that no American publishing house would bring out a work in the German language with such faults in usage, spelling, and construction as disfigure the work under consideration. The reviewer makes no attempt to list these errors.

There is no better evidence of the uncertainty which attaches to the ancient mathematics of China than the fact that Mr. Mikami's description of the *Arithmetic in Nine Sections* is entirely different (footnote, p. 10) from the description given by T. Hayashi in his *Brief History of Japanese Mathematics* (Nieuw Archief, Tweede Reeks, Deel VI, 306–307, not accessible to me). However this uncertainty is also characteristic of the history of ancient Hindu mathematics.

One of the most striking chapters of the work (Chap. IV) deals with *The Arithmetical Classic of Sun-Tsu.* The operations of multiplication and division and extraction of square root correspond almost precisely, mechanically, to these operations as taught in the early treatises explaining the Hindu art of reckoning. Thus in multiplication the unit of the lower number is placed below the highest digit of the upper number and the highest digit of the upper number is then multiplied by the lower number, the product being arranged in a line between the

two. Commonly the partial products in the treatises based upon the arithmetic of Al-Khowarizmi are placed above the two lines of multiplier and multiplicand, or on the upper line, this number being deleted in the course of the operation. Then the lower number is drawn back one digit and the work proceeds as before both in the Chinese work and in the European, which is based upon the Arabic and Hindu. The correspondence seems too striking to be wholly accidental but the connection has not yet been established. These calculations were effected doubtless with the *sangis* or calculating pieces upon some sort of abacus. The date of the treatise is left undetermined. The mathematician, Tai Cheng (1722–1777), maintains that the composition of Sun-Tsu's treatise could not ante-date the reign of the Han emperor, Ming-Ti (first century A.D.). Apparently Mr. Mikami construes this to mean that this work was composed about the first century of the Christian era, but no better evidence than the uncertain statement I have quoted is presented as to the age of the treatise. Somewhat similar treatises on the fundamental operations are found in the sixth century in China.

The Tsu Ch'ung-chih (fifth century) approximation for the value of π, namely 355/113, which is easily remembered from the sequence 1 1 3 3 5 5, should be familiar to every teacher of elementary mathematics and this might well be designated as the Chinese approximation if the name "Tsu Ch'ung-chih's approximation" seems too formidable. Other points particularly worthy of note include the *t'ai-yen* method for the solution of indeterminate equations, the *celestial element* process for the treatment of numerical equations, "Pascal's triangle" in China in the early fourteenth century, and Seki's development of the determinant idea in the seventeenth century. One *t'ai-yen* problem proposed by Sun-Tsu is to find a number which divided by 3, 5, and 7 respectively gives as remainders 2, 3, and 2 respectively. This is of particular interest because the same problem with the exception of the third remainder, 4 replacing 2, given by Leonard of Pisa in 1202, bears witness to contact between East and West. The thirteenth century studies by Ch'in Chiu-shao on approximation of the roots of numerical equations correspond closely to the methods of Newton and Horner.

Numerical errors are somewhat common in the work, as for example: 293 for 233 (p. 32); 30 9/60 for 36 9/10 (p. 54); 1337 1/20 for 397 3/4 (p. 54). The problem of the 10 foot bamboo which is broken at such a point that it touches the ground 3 feet from the stem (p. 23) is not found in Brahmagupta's work but twice in Bhaskara (Colebrooke, *Algebra . . . from the Sanskrit of Brahmagupta and Bhâscara*, London, 1817, pp. 203–204 and 64–65). There appears to be no point in stating that there is "no use of decimals" (p. 12) in these ancient works, for decimal subdivisions abound and decimal fractions as we have them, using the decimal point, would be most unlikely to occur. The explanation (p. 2) of the *chia-tsu* or sexagesimal system of numeration is not at all clear, and the notation requires ten numeral words in the first series whereas only nine are given.

Mikami compares Seki with Newton (p. 158). He speaks of "the Japanese Newton," and says "If Seki did not surpass Newton in his achievements, yet he was no inferior of the two" (Galileo being the other to whom this statement

refers). The reader will look in vain in Mr. Mikami's work for any justification of this effusive praise which flatters neither Seki, nor Newton, nor the Japanese. Seki was undoubtedly for his time and place an able mathematician, but the luster of his light is only dimmed by comparison with Newton or Galileo.

In concluding our review we wish to state again that Mr. Mikami has rendered a real service to the history of science by his exposition of the development of mathematics in China and Japan.

L. C. KARPINSKI.

PROBLEMS AND SOLUTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

SPECIAL NOTICE. In proposing problems and in preparing solutions, contributors will please follow the form established by the MONTHLY, as indicated on the following pages.

In particular, a solution should be preceded by the number of the problem, the name and address of the proposer, the statement of the problem, and the name and address of the solver.

The solution should then be given with careful attention to legibility, accuracy, brevity without obscurity, paragraphing and spacing, having in mind the form in which it will appear on the printed page.

Please use paper of letter size, write on one side only, leaving ample margins, put one solution only on a single sheet and include only such matter as is intended for publication.

Drawings must be made *clearly* and *accurately* and an extra copy furnished on a *separate sheet* ready for the engraver.

Unless these directions are observed by contributors, solutions must be entirely rewritten by the committee or else rejected.

Selections for this department are made two months in advance of publication.

Please send all solutions direct to the chairman of the committee.

MANAGING EDITOR.

ALGEBRA.

When this issue was made up, solutions had been received for 403–4–5–7–9 and 410. Solutions of 406 and 408 are desired.

413. Proposed by C. N. SCHMALL, New York City.

Apply Euler's transformation to show that

$$1 + 2^2x + 3^2x^2 + 4^2x^3 + 5^2x^4 + \cdots = \frac{1+x}{(1-x)^3}.$$

(BROMWICH, *Theory of Infinite Series*, p. 62, ex. 20.)

414. Proposed by R. D. CARMICHAEL, Indiana University.

Prove by means of an example that one of the series

$$\sum_{k=1}^{\infty} \frac{1}{c_k}, \qquad \sum_{k=1}^{\infty} \frac{1}{c_k - 1}, \qquad c_k \neq 0, 1,$$

may be divergent while the other is convergent.

415. Proposed by C. N. SCHMALL, New York City.

Show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.$$

(BROMWICH, *Infinite Series*, p. 187; and also p. 452, ex. 7, (iii)).

416. Proposed by C. E. FLANAGAN, Wheeling, Va.

The sides of a given rectangle are a and b, in which a rectangle is to be inscribed one of whose sides is c. Find the other side, using Euler's rule for quartics.

417. Proposed by A. J. RICHARDSON, Marquette, Mich.

Required to reduce the quartic

$$x^4 + px^2 + qx + r = 0$$

to the form

$$(x^2 + k)^2 = [(2k - p)x + (2k - q)]^2,$$

wherein k is the solution of a certain cubic. Hence, express the solution of the given quartic in terms of p, q, r, and k.

GEOMETRY.

When this issue was made up, solutions had been received for 434–5–6 and 439. Solutions of 432–3 and 441–2–3 are desired.

441. Proposed by H. E. TREFETHEN, Colby College.

In the triangle ABC find the locus of all points at which the sides AB, AC subtend equal angles.

442. Proposed by J. B. SMITH, Hampden-Sidney, Va.

If any three straight lines, AD, BE, CF, be drawn from the corners of the triangle ABC to the opposite sides, a, b, c, they will enclose an area. If Δ, Δ'' be the areas of the triangles ABC, DEF show that

$$\frac{\Delta''}{\Delta} = \frac{(AF \cdot BD \cdot CE - AE \cdot CD \cdot BF)^2}{(ab - CE \cdot CD)(bc - AE \cdot AF)(ca - BF \cdot BD)},$$

where the signs of the factors are to be determined by the following rule: Each segment being measured from one of the corners of the triangle ABC, along one of the sides, is to be regarded as positive or negative according as it is drawn towards or from the other corner in that side.

443. Proposed by C. N. SCHMALL, New York City.

A quadrilateral of any shape whatever is divided by a transversal into two quadrilaterals. The diagonals of the original figure and those of the two resulting (smaller) figures are then drawn. Show that their three points of intersection are collinear.

CALCULUS.

When this issue was made up, solutions had been received for 346–7–9 and 351–4–6–7. Solutions of 332–5, 340–5–8 and 352–3–5 are desired. A complete solution of 339 is also desired.

363. Proposed by B. F. FINKEL, Drury College.

The axis of a right prism whose cross-section is a regular polygon of n sides coincides with the diameter of a sphere of radius R. Find the surface of the sphere included within the prism.

364. Proposed by EMMA GIBSON, Drury College.

Solve the differential equation

$$(xp - y)^2 = a(1 + p^2)(x^2 + y^2)^{3/2}, \qquad \text{where } p = dy/dx.$$

365. Proposed by C. N. SCHMALL, New York City.

Show that the area inclosed by each of the following three curves is equal to that of the circle of radius a; viz., πa^2.

(1) $$a^2x^2 = y^3(2a - y),$$

(2) $$a^2 - x^2 = (y - mx^2)^2,$$

(3) $$(xy + c + bx^2)^2 = x^2(a^2 - x^2).$$

MECHANICS.

When this issue was made up, solutions had been received for 285–8–9 and 292. Solutions of 266, 268, 269, 271 to 275, inclusive, and 277 to 279 inclusive are desired.

293. Proposed by B. F. FINKEL, Drury College.

A man of weight W stands on smooth ice; prove that if, when he gradually parts his legs, kept straight, with his feet in contact with the ice, the pressure of his feet on the ice be constant, his head will descend with uniform acceleration; and that, if f be the acceleration of his head, when his feet exert no pressure on the ice, their pressure on the ice, were f' the acceleration of his head, would be equal to $\frac{f-f'}{f}W$.

(From WALTON'S *Problems in Theoretical Mechanics.*)

294. Proposed by EMMA GIBSON, Drury College.

A sphere, revolving about a diameter and not acted on by any extraneous force, expands symmetrically; prove that its vis viva varies inversely as its moment of inertia about a diameter.

NUMBER THEORY.

When this issue was made up, solutions had been received for 206–7, 210, 213. Solutions of 192 and 196 are desired.

215. Proposed by R. D. CARMICHAEL, Indiana University.

Find one or more values of n such that a polygon of n sides shall have the number of its diagonals equal to the cube of an integer.

216. Proposed by ELIJAH SWIFT, Princeton, N. J.

If p is a prime > 3, show that $\Sigma_{a=1}^{a=p-1} 1/a \equiv 0 \pmod{p^2}$, where $1/a \equiv x$, if $ax \equiv 1 \pmod{p^2}$

217. Proposed by E. T. BELL, Seattle, Wash.

(i) If r is a prime greater than 2, and $p \equiv 2^a r + 1$ is prime, the only solution, when n is greater than 2, of $x^n - y^n = p$, is $n = 3$, $x = 2$, $y = 1$.

(ii) The only primes that are simultaneously of the forms $4k + 1$ and $3^m - 2^m$ are 1 and 5.

(iii) Generalize (ii).

SOLUTIONS OF PROBLEMS.

ALGEBRA.

385. Proposed by J. F. LAWRENCE, Stillwater, Oklahoma.

Show that if p is prime and > 3, $(2p)! - 2 \cdot p!p!$ is divisible by p^5.

Solution by Elijah Swift, Princeton, N. J.

Dividing the above expression by $2p \cdot p!$ we have to show that

$$(1) \qquad (2p-1)(2p-2)(2p-3)\cdots(2p-(p-1)) - (p-1)!$$

is divisible by p^3.

Consider the polynomial $P(x) = (x-1)(x-2)(x-3)\cdots(x-p+1)$.[1] If this is expanded in powers of x, we obtain an expression of the form

$$x^{p-1} - A_1x^{p-2} + A_2x^{p-3} + \cdots + A_{p-3}x^2 - A_{p-2}x + A_{p-1},$$

[1] See BACHMANN, *Niedere Zahlentheorie*, p. 155. I have changed signs, but evidently the same results hold.

where

$$A_1 = 1 + 2 + 3 + \cdots + p - 1$$
$$A_2 = 1\cdot 2 + 1\cdot 3 + \cdots + 2\cdot 3 + \cdots + (p-2)(p-1)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$A_{p-1} = 1\cdot 2\cdot 3\cdots(p-1).$$

It is proved by Bachmann that each of these A's, except A_{p-1}, is divisible by p.

Obviously, from the expression of $P(x)$ in factor form $P(p) = (p-1)!$ It follows that $P(p) - (p-1)! = p^{p-1} - A_1p^{p-2} + \cdots + A_{p-3}p^2 - A_{p-2}p = 0$. If then A_{p-3} is divisible by p, A_{p-2} will be divisible by p^2. If $p = 3$, however, $A_{p-3} = 1$, or rather there is no A_{p-3}, and our reasoning breaks down.

Now expand (1). We shall evidently have $P(2p) - (p-1)! = (2p)^{p-1} - A_1(2p)^{p-2} + \cdots + A_{p-3}(2p)^2 - A_{p-2}(2p)$. But A_{p-3} is divisible by p, and A_{p-2} by p^2. Therefore, (1) is divisible by p^3, and $(2p)! - 2\cdot p!p!$ by p^5.

398. Proposed by R. D. CARMICHAEL, Indiana University.

In the equation $x^3 + \alpha x + \beta = 0$, α is an integer divisible by p^2 and β is an integer divisible by p, p being a prime number. Prove that β is divisible by p^3 if the equation is reducible.

SOLUTION BY ELMER SCHUYLER, Brooklyn, New York.

Let $(x^2 + lx + m)(x + n) \equiv x^3 + \alpha x + \beta$. Then since $\alpha = p^2t$ and $\beta = pr$, $l = -n$; $m - n^2 = p^2t$; $mn = pr$. Hence, $m = pr/n$ and $pr/n - n^2 = p^2t$, or $pr - n^3 = p^2nt$. Consequently, n is divisible by p and $n = ps$, since n clearly is an integer. Whence $pr - p^3s^3 = p^3st$ or $r/p^2 = st - s^3$. Hence, we have $\beta = p^3(st - s^3)$.

Also solved by B. LIBBY, G. W. HARTWELL, A. M. HARDING, and ELIJAH SWIFT.

399. Proposed by W. H. BUSSEY, University of Minnesota.

A borrows from B \$1,500 and pays back \$34 a month for 63 months. If the last payment closes the account, what rate of interest has A been paying.

SOLUTION BY GEO. W. HARTWELL, Hamline University.

This problem is equivalent to the following: B pays A \$1,500 for an annuity of \$34 per month to run 63 months. What is the rate realized?

Let $r =$ the annual rate. Then $r/12 =$ the monthly rate. Hence,

$$34\left(1+\frac{r}{12}\right)^{62} + 34\left(1+\frac{r}{12}\right)^{61} + \cdots + 34 \equiv 34\left[\frac{\left(1+\frac{r}{12}\right)^{63} - 1}{\frac{r}{12}}\right] = 1{,}500,$$

or the total number of dollars realized.

Solving this equation by the usual method, we find $r = 14.31$ per cent.

Also solved by G. Y. SASNOW, ALBERT R. NAUER, WILLIAM CULLUM, W. C. EELLS, and CLIFFORD N. MILLS.

400. Proposed by C. N. SCHMALL, New York City.

Sum the series $1 + 2x + 3x^2 + 4x^3 + \cdots$.
(BROMWICH, *Infinite Series*, p. 129, ex. 1.)

I. SOLUTION BY GEO. W. HARTWELL, Hamline University.

Let

$$S = 1 + 2x + 3x^2 + 4x^3 + \cdots.$$

Then

$$(1 - x)S = 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1 - x}, \text{ for } -1 < x < +1.$$

Hence

$$S = \frac{1}{(1 - x)^2}.$$

II. SOLUTION BY ELIJAH SWIFT, Princeton University.

The series converges for $-1 < x < +1$ and may be integrated in the open interval -1 to $+1$.

Then $\int_0^x S(x)dx = x + x^2 + x^3 + \cdots$, which is equal to $\frac{x}{1 - x}$. Differentiating this, we obtain $\frac{1}{(1 - x)^2}$ as the sum of the given series.

III. SOLUTION BY J. BROOKS SMITH, Hampden Sidney, Va.

The following solution is given by CHARLES SMITH, *Treatise on Algebra*, ex. 1, p. 413.

$$S_{n+1} = 1 + 2x + 3x^2 + 4x^3 + \cdots + (n + 1)x^n,$$

$$(1 - x)^2 = 1 - 2x + x^2.$$

Hence,

$$(1 - x)^2 \cdot S_{n+1} = 1 + x^{n+1}[n - 2(n + 1)] + (n + 1)x^{n+2}$$

(all the other terms vanishing on account of the identity,

$$k - 2(k - 1) + k - 2 \equiv 0).$$

Hence

$$(1 - x)^2 S_{n+1} = 1 - (n + 2)x^{n+1} + (n + 1)x^{n+2};$$

whence

$$S_{n+1} = \frac{1}{(1 - x)^2} - \frac{(n + 2)x^{n+1} - (n + 1)x^{n+2}}{(1 - x)^2}.$$

When $n \doteq \infty$, $S_{n+1} \doteq S \doteq \frac{1}{(1 - x)^2}$ for $-1 < x < +1$.

Solved in various other ways by ELMER SCHUYLER, B. LIBBY, HORACE OLSON, F. M. MORGAN, A. L. MCCARTY, C. HORNUNG, A. M. HARDING, CLIFFORD N. MILLS, H. C. FEEMSTER, OSCAR SCHMIEDEL, and A. G. CARIS.

A solution of 396 was received from ELIJAH SWIFT too late for credit in the last issue.

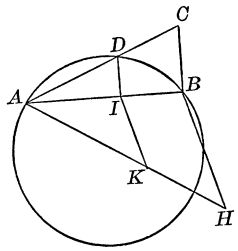
GEOMETRY.

428. Proposed by R. D. CARMICHAEL, Indiana University.

On a given chord of a circle as a base construct a right triangle with vertex outside of the circle such that its hypotenuse shall be bisected by its point of intersection with the circle. Are ruler and compasses sufficient to construct a triangle whose hypotenuse shall be thus divided in any ratio whatever?

SOLUTION BY EMMA M. GIBSON, Drury College.

The ruler and compasses are sufficient to construct a triangle whose hypotenuse shall be divided in any ratio as $m : n$, according to the conditions of the problem.



For let AB be the given chord. From A draw any line as AH and from A lay off $AK = m$ and $KH = n$. Then connect B and H and draw KI parallel to HB and intersecting AB in I. At I erect a perpendicular to AB intersecting the circle in D. Join A and D by a straight line and produce it until it meets a perpendicular erected at B in C. Then the right triangle ABC has its hypotenuse, AC, divided at D in the ratio of $m : n$.

For in the similar triangles AIK and ABH, $AI : IB = m : n$ and in the similar triangles ABC and AID, $AI : IB = AD : DC$. Hence, $AD : DC = m : n$.

When $m = n$, D bisects the hypotenuse.

Also solved by CLIFFORD N. MILLS, F. M. MORGAN, C. HORNUNG, HORACE OLSON, and S. W. REAVES.

429. Proposed by JOHN A. BIGBEE, Little Rock, Ark.

In the trihedral angle V-ABC, the face angle AVB is bisected by the straight line VD. Is it true that the angle DVC is less than, equal to, or greater than, half the sum of the angles AVC and BVC, according as $\angle CVD$ is less than, equal to, or greater than 90°?

SOLUTION BY A. M. HARDING, University of Arkansas.

(1) $\angle CVD < 90°$. (Fig. 1.) Take $VA = VB$. Join A and B. Let this line cut the bisector of $\angle AVB$ at D. Through D pass a plane perpendicular to

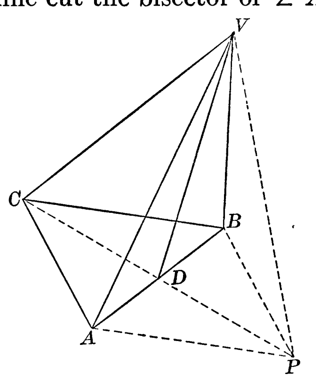


FIG. 1.

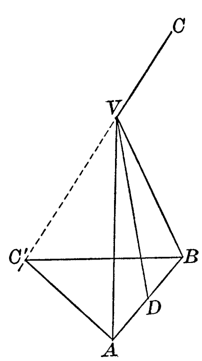


FIG. 2.

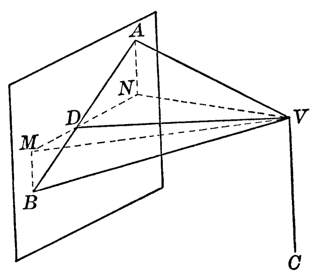


FIG. 3.

VD cutting VC at C. Produce CD to P making $DC = DP$. Draw the lines AP, BP, and VP. From this construction it easily follows that $\angle BVC = \angle PVA$ and $\angle CVP = 2\angle CVD$. But $\angle CVP < \angle AVC + \angle PVA$. Hence $\angle CVD < \frac{1}{2}(\angle AVC + \angle BVC)$.

(2) $\angle CVD > 90°$. (Fig. 2.) Produce CV through V and draw plane ABC' as in (1). Then $2\angle C'VD < \angle AVC' + \angle BVC'$ by (1). Hence $2(180° - \angle C'VD) > 180° - \angle AVC' + 180° - \angle BVC'$ or $\angle CVD > \frac{1}{2}(\angle AVC + \angle BVC)$.

(3) $\angle CVD = 90°$. (Fig. 3.) In this case the plane through AB is parallel to VC. Draw a plane MNV through VD perpendicular to VC and cut by planes BVC and AVC in the lines MV and NV respectively. It can be easily shown that $\angle MVB = \angle NVA$. Hence

$$\angle AVC + \angle BVC = 180°$$

and

$$\angle CVD = \tfrac{1}{2}(\angle AVC + \angle BVC).$$

Also solved by B. Libby, F. M. Morgan, and Geo. W. Hartwell.

CALCULUS.

338. Proposed by RICHARD LOCHNER, Philadelphia, Pa.

An elliptical field has a major axis of 100 feet and a minor axis of 10 feet. A cow is tethered at the end of the major axis and another at the end of the minor axis. If each cow can graze over half the field, how long is the rope of each? What is the area of the portion over which the cows can graze in common?

Solution by B. F. Finkel, Drury College.

The central equation of the elliptic field is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Let r_1 be the length of rope by which one cow is tethered at the point A_1, the right-hand extremity of the major axis. The equation of the circle over which this cow can browse is $(x - a)^2 + y^2 = r^2$. The coördinates of the point of intersection, P_1, of this circle with the ellipse are

$$(x_1, y_1) \equiv \left(a\frac{[a^2 - \sqrt{b^4 + r^2(a^2 - b^2)}]}{(a^2 - b^2)}, \right.$$

$$\left. \sqrt{2a^4\frac{\sqrt{b^4 + r^2(a^2 - b^2)} - b^2(a^2 - b^2)r^2 - a^2(a^4 + b^4)}{(a^2 - b^2)}} \right),$$

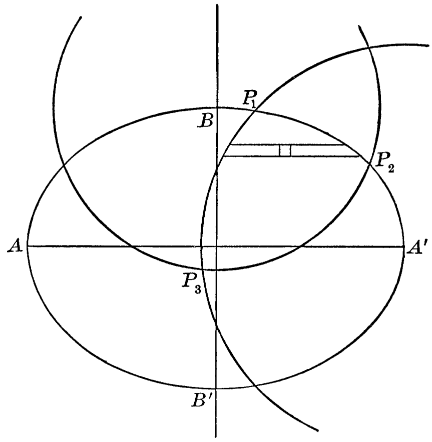
The area common to the ellipse and the circle is

$$\text{area} = 2\int_0^{y_1}\int_{a-\sqrt{r^2-y^2}}^{(a/b)\sqrt{b^2-y^2}} dxdy$$

$$= 2\left[\frac{a}{b}\frac{y_1}{2}\sqrt{b^2 - y_1^2} + \frac{b^2}{2}\sin^{-1}\frac{y'}{b} - ay' + \frac{y'}{2}\sqrt{r^2 - y_1^2} + \frac{r^2}{2}\sin^{-1}\frac{y_1}{r}\right],$$

which, by the conditions of the problem, $= \frac{1}{2}\pi ab$.

Substituting the value of y_1 in this equation, we have an equation in r. At the cost of great labor, this equation can be solved, by trial, to any degree of accuracy.



In a similar manner, the area over which a cow tethered at B can graze, may be found. Let the coördinates of the points of intersection, P_2, be (x_2, y_2). The length of the radius, BP_2, may be found to any desired degree of accuracy, as above. When this is found x_2 and y_2 become known.

Then to find the area common to the two circles and the ellipse we have only to perform the following integration,

$$\int_{y_3}^{y_2}\int_{a-\sqrt{r_1^2-y^2}}^{\sqrt{r_2^2-(y-b)^2}} dxdy + \int_{y_2}^{y_1}\int_{a-\sqrt{r_1^2-y^2}}^{(a/b)\sqrt{b^2-y^2}} dxdy,$$

where y_3 is the ordinate of P_3.

The case when the circle whose center is B cuts the ellipse in four points, would require still more labor.

343. Proposed by C. N. SCHMALL, New York City.

Show that the envelope of the system of circles whose radii are the ordinates of an ellipse is a concentric ellipse having the same minor axis as the given ellipse.

SOLUTION BY I. A. BARRETT and F. C. REISLER, University of Chicago.

Choose the axes so that the equation of the ellipse is in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Since the radii of the circles are to be ordinates of the ellipse, we have

$$(X-x)^2+Y^2=\frac{b^2}{a^2}(a^2-x^2), \tag{1}$$

where (X, Y) are running coördinates of the circle.

Differentiating (1) with respect to x, we have

$$X - x = \frac{b^2}{a^2}x. \tag{2}$$

Solving (1) and (2) simultaneously, we obtain

$$X^2\left[1 - \frac{a^2}{a^2 + b^2}\right]^2 + Y^2 = b^2 - \frac{b^2a^2}{(a^2 + b^2)^2}X^2.$$

Reducing, we have

$$\frac{X^2}{a^2 + b^2} + \frac{Y^2}{b^2} = 1.$$

This is an ellipse with same minor axis as original ellipse.

Also solved by FRANK C. MOORE, H. C. FEEMSTER, and the PROPOSER.

MECHANICS.

281. Proposed by C. N. SCHMALL, New York City.

ABC is a triangle inscribed in a circle, center O, and L, M, N, are the centers of gravity of the sectors AOB, BOC, COA. Show that

$$\frac{AB}{OL} + \frac{BC}{OM} + \frac{CA}{ON} = 3\pi.$$

SOLUTION BY S. W. REAVES, University of Oklahoma.

The well-known formula for the center of gravity of a sector of a circle gives

$$OL = \frac{4}{3} \cdot \frac{r \sin \frac{1}{2}AOB}{\text{angle } AOB} = \frac{AB}{3 \angle C}.$$

Hence $\dfrac{AB}{OL} = 3 \angle C$. Similarly, $\dfrac{BC}{OM} = 3 \angle A$, and $\dfrac{CA}{ON} = 3 \angle B$. Adding,

$$\frac{AB}{OL} + \frac{BC}{OM} + \frac{CA}{ON} = 3(\angle A + \angle B + \angle C) = 3\pi.$$

Also solved by A. M. HARDING, CHARLES E. HORNE, P. PEÑALVER, B. LIBBY, ELMER SCHUYLER, WALTER C. EELLS, RICHARD MORRIS, H. C. FEEMSTER, J. B. SMITH, J. W. COLSON, F. C. REISLER, and I. A. BARRET.

282. Proposed by R. P. LOCHNER, Philadelphia, Pa.

A car weighing 10 tons (2,240 lbs. each) attains a speed of 15 miles an hour from rest in 24 seconds, during which it covers 100 yards. If the space-average of the resistances is 30 lbs. per ton, find the average horse-power used to drive the car. (MORLEY'S *Mechanics for Engineers*, p. 66.)

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

Force (lbs.) required if there were no friction $= \dfrac{m}{g} \cdot \dfrac{v}{t} = \dfrac{2240}{32} \times \dfrac{22}{24} = 641$ lbs. approximately.

Force (lbs.) required to overcome friction = 300 lbs. Total force acting is therefore 941 lbs.

$$\text{Power applied} = \frac{Fs}{t} = \frac{941 \times 300}{24} = 11{,}762 \text{ ft. lbs. per sec.} = 21 \text{ H.P. (approx.).}$$

283. Proposed by C. N. SCHMALL, New York City.

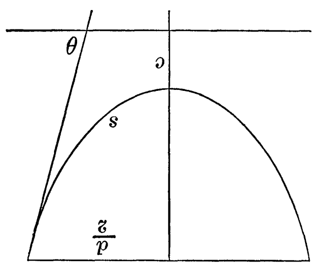
The maximum length of a certain chain which can be suspended from one end without breaking is l. It is desired to form a catenary with a length $2l/k$ of the chain, the points of support being a distance d apart, in the same horizontal line. Show that the maximum value of d is

$$\frac{2l}{k}(k^2 - 1)^{1/2} \log \left(\frac{k+1}{k-1}\right)^{1/2}.$$

SOLUTION BY A. M. HARDING, University of Arkansas.

Let w = weight per unit length. Then wl = maximum tension the chain will stand.

The tension at the point of support is given by $T \sin \theta = ws$ where s = one-half the length of the chain and θ is the angle that the tangent to the catenary at that point makes with the x-axis.



If we put $T = wl$ and $s = l/k$ we find $\sin \theta = 1/k$. But $\tan \theta = \frac{1}{2}(e^{d/2c} - e^{-(d/2c)})$ where c is the distance along the Y-axis from the origin to the catenary.

Hence we have

$$\tfrac{1}{2}(e^{d/2c} - e^{-(d/2c)}) = \frac{1}{\sqrt{k^2 - 1}}.$$

From this equation we obtain

$$d = 2c \log \left(\frac{k+1}{k-1}\right)^{\frac{1}{2}}.$$

We have also the intrinsic equation of the catenary $s = c \tan \theta$, from which we obtain

$$c = \frac{l}{k}\sqrt{k^2 - 1}.$$

Whence

$$d = \frac{2l}{k}(k^2 - 1)^{\frac{1}{2}} \log \left(\frac{k+1}{k-1}\right)^{\frac{1}{2}}.$$

Also solved by J. W. COLSON.

NUMBER THEORY.

189. Proposed by V. M. SPUNAR, Chicago, Illinois.

If p and $p_1 = 2^p - 1$ are primes, then are the numbers of the sequence $p_1 = 2^p - 1$, $p_2 = 2^{p_1} - 1$, $p_3 = 2^{p_2} - 1$, $\cdots$, $p_n = 2^{p_{n-1}} - 1$ all primes?

REMARKS BY R. D. CARMICHAEL, Indiana University.

If the conjecture stated in this problem is true it is highly desirable to have a proof of it. It would be a significant contribution to the theory of prime numbers; for the theorem so obtained would afford a means of recursion by which a sequence of indefinitely increasing prime numbers could be written out consecutively. No such tool has yet been invented. Euler and Legendre sought in vain for analytical expressions which would serve just this purpose. Fermat believed, though he confessed that he was unable to prove, that he had found such an analytical expression in

$$2^{2n} + 1.$$

Euler pointed out the error of this opinion by showing that 641 is a factor of this number for the case $n = 5$.

From these historical facts one is led to suppose that the proof of the conjecture of this problem is probably difficult, in case the conjecture is true; if the conjecture is false, that would be shown by means of an example. The numbers to be dealt with, however, are so large that the construction of such an example (in case it exists) would perhaps be very tedious.

Note. A number of incorrect solutions of problem 198 have been received. Will our contributors please give it a more careful consideration? It is very probably not easy to solve.

EDITORS.

199. Proposed by R. P. LOCHNER, Philadelphia, Pa.

Find three integral squares, such that the sum of every two of them shall be a square.—Alsop's Algebra, 3d edition (Philadelphia, 1859), p. 296, Ex. 13.

SOLUTION BY ARTEMAS MARTIN, Washington, D. C.

Let x^2, y^2 and z^2 be the required squares; then we must have

$$x^2 + y^2 = \square,\ x^2 + z^2 = \square,\ y^2 + z^2 = \square, \tag{1, 2, 3}$$

Assume $y = \dfrac{(m^2 - n^2)x}{2mn}$, $z = \dfrac{(p^2 - n^2)x}{2pn}$; then (1) and (2) are satisfied and (3) becomes, after striking out common square factors,

$$p^2(m^2 - n^2)^2 + m^2(p^2 - n^2)^2 = \square, \tag{4}$$

which may be otherwise written

$$m^2p^2(m^2 + p^2) - 4m^2n^2p^2 + n^4(m^2 + p^2) = \square; \tag{5}$$

and (5) will be a square when

$$m^2 + p^2 = 4n^2. \tag{6}$$

Put $m = \dfrac{2n(r^2 - s^2)}{r^2 + s^2}$, $p = \dfrac{4nrs}{r^2 + s^2}$; then (6) is satisfied. Take $r = 2$, $s = 1$, and we have $m = \dfrac{6n}{5}$, $p = \dfrac{8n}{5}$; $y = \dfrac{11x}{60}$, $z = \dfrac{39x}{80}$. Now take $x = 240$ and we have $y = 44$, $z = 117$, the least numbers known.

The numbers given by Alsop (528, 5796, 6325) are much larger, and same as found by Bonnycastle in his Introduction to Algebra published in London more than 100 years ago.

For other solutions, see *Mathematical Magazine*, Vol. II, No. 11 (Dec., 1898), pp. 214–16; also, the *Normal Monthly*, Vol. III, No. 10 (Millersville, Pa., June, 1876), p. 119.

Also solved by C. E. Githens.

MISCELLANEOUS QUESTIONS.

Edited by R. D. Carmichael.

QUESTION.

11. In our courses of study is it desirable to give more consideration to vector analysis? What topics should be included in a first treatment of this subject?

REPLY.

6. In what ways and to what extent will the teaching of mathematics and the content of the curriculum probably be affected by the increasing demand for vocational training?

I. Remarks by W. T. Stratton, Kansas State Agricultural College, Manhattan, Kansas.

There is a demand today for the practical in education. Every subject that is to remain in our curriculum must do so by showing that it has a real place in the lives of the people. The demand for vocational training will affect both the teaching of mathematics and the content of the curriculum to a very marked degree.

Every new teacher in our high schools has a tendency to teach the topics that were taught to him, and in the same way as he was instructed. Only the unusual man will break away from this practice. So the change in teaching and the vocational attitude toward the subject must find a place in the colleges and universities before they will permeate into all the schools. As long as the former turn out men and women grounded in the belief that the chief reason for the study of mathematics is in its disciplinary value for the intellect the high schools will in general follow the old traditional course. The change in teaching, it seems to me, will come principally from a change of attitude of the teachers toward the subject. The teachers will come to look upon the subject from the social point of view, that is, they will not only teach the mathematical computations and methods of argumentation, but they will also give the subject a social setting; they will show to the students the place in society at which the particular topic under consideration has its direct applications.

The kind of training college men and women have had in the past has not fitted them for work in the vocational schools. Until the normal schools and colleges give vocational subjects, and train the teachers from the vocational point of view, nothing but poorly taught vocational mathematics can be expected in the high schools. I do not mean that the class room instruction will be essentially changed, but that there will be a closer correlation of the mathematics

work with that of the other departments, and that the pupils will be brought into contact with real problems and not the mere statements of problems. Nor do I believe that it will ever be practicable in high school mathematics for teachers to take their classes into the laboratories or workshops and there work on a series of experiments so planned as to present the problems which the pupils will then be required to solve. It appears that there are many possibilities along this line that have never been realized, but it does not seem that one of the chief aims in mathematical instruction can best be served in this way. Mathematics consists of a system of knowledge which demands logical thinking for its mastery; and the number and variety of problems that will come up in this laboratory work will not be great enough to give the requisite experience in thinking.

The advent of vocational training into the secondary schools will probably bring about a closer study of the needs of the pupils. They will not be required to take a year and a half of algebra and a course in plane and solid geometry irrespective of the length of time they will likely spend in school or of the vocations for which they are preparing. Efforts are being put forth to make the courses such that if it becomes necessary for the pupils to leave school at the close of any term their work in mathematics will be of such a nature as to be of value to them. The amount and the character of the mathematics for students who are expecting to attend college will likely not be lessened. In the larger industrial centers the tendency will be to widen the scope of subjects, that is, along with the traditional algebra and geometry the most practical parts of trigonometry and a course in applied mathematics summarizing and vitalizing the subjects learned will make up the course. In the agricultural high schools the course in mathematics will likely be shortened and a course in applied arithmetic and courses in algebra and geometry, which emphasize the most practical phases of interest to the farmer, will replace the present courses. The course for the girls in the different schools will also probably be shortened, and the subjects of arithmetic, algebra, and geometry will be treated along with the applications of a kind that will be of most interest and value to them.

The high school in all its courses will probably give more consideration to the special needs of the pupils than to their preparation for college. The colleges will require fifteen units of work with very few limitations. Thus the small schools will be free to offer courses best fitted for the students in their particular localities and at the same time give the training requisite for admission to college.

NOTES AND NEWS.

UNDER THE DIRECTION OF FLORIAN CAJORI.

In H. Weber's *Kleines Lehrbuch der Algebra*, 1912, the same proof of the simplicity of the alternating group of degree n, $n \neq 4$, appears twice, on pages 215 and 373 respectively.

The lectures on recent advances in physics, delivered by Professor Volterra at Clark University in September, 1909, were published in the double number of the *Archiv der Mathematik und Physik* which appeared February 17, 1914.

It is proposed to erect a monument in memory of P. S. Laplace at his birthplace, Beaumont-en-Auge, France. The money to defray the expense is to be raised by international subscription. The firm, M. Gauthier-Villars, Paris, France, has offered to receive subscriptions for this purpose.

The part of the *Encyclopédie des Sciences Mathématiques* dated December 23, 1913, is devoted to Projective Geometry (143 pages) and to Configurations (17 pages). Among the new parts announced in press is the rest of the article on Calculus of Variations, and an article on the Developments of Hydrodynamics. Tribune publique 23 is inserted in the present part.

Florian Cajori, of Colorado College, will attend the Napier Tercentenary Celebration next July in Edinburgh, and will present a historical paper on that occasion. He will spend the summer in Great Britain and later travel on the Continent. In the fall of 1915 he will resume his work at Colorado College.

The January, 1914, issue of the *Irish Journal of Education* contains a further instalment of the Report of the National (American) Committee of Fifteen on Geometry Syllabus, embracing that part of the historical introduction which refers to England. The whole report is to be printed in the *Journal* by permission of the Committee.

Professor L. C. Karpinski, of the University of Michigan, held an Institute on Arithmetic before the Toledo, Ohio, Teachers' Association during the week of March 2–7, 1914. His lectures were mostly to the grade teachers, but one was given to the high school teachers. His underlying purpose was to emphasize the practical side of arithmetic.

The courses in mathematics to be given at the summer session of the University of Michigan, June 26 to August 21, 1914, include plane geometry, solid geometry, elementary algebra, trigonometry, college algebra, analytic geometry, geometry and algebra (teachers' course), calculus (two courses), theory of finance, insurance, and statistics, history of mathematics, differential equations, projective geometry, theory of functions, infinite series and products, the theory of potential, advanced calculus. The members of the regular staff offering these courses are professors BEMAN, MARKLEY, and FORD, assistant professors BRADSHAW and KARPINSKI, and instructors POOR, HILDEBRANDT, FORSYTH, and MILLER.

The following courses in mathematics will be offered during the Summer session of Dartmouth College, July 6 to August 15, 1914: In addition to elementary college courses,—by Professor PITCHER, principles of calculus; by Professor BILL, solid analytic geometry, advanced aspects of elementary geometry (the latter intended primarily for teachers); by Dr. MORGAN, higher geometry, the teaching of elementary algebra.

The University of Illinois announces a course of lectures on General Analysis by Professor Fréchet of the University of Poitiers, France. The course will

consist of four lectures per week throughout both semesters of 1914–'15. Professor Fréchet is recognized as one of the leaders in this field of mathematics and through his publications has contributed largely to its literature.

The Illinois chapter of the Society of Sigma Xi held a regular meeting on April 15 at which Mr. A. O. MacGillivray spoke on "Spiders and their spinning work."

The Spring quarter gathering of the Society of Sigma Xi at the University of Chicago was held on Tuesday, May 19. The after-dinner speaker was Dr. Ira Remsen, President *Emeritus* of Johns Hopkins University, whose subject was "Reminiscences of Sylvester and Rowland."

The Harvard University Press announces the early publication of "The Hyperbolic Functions of Complex Variables" by Professor A. E. Kennelly.

Professor David Eugene Smith, of the Teachers College, Columbia University, will attend the Napier tercentenary celebration at Edinburgh, next July, and will present a historical paper on that occasion.

Professor H. Andoyer, of the University of Paris, announces that he has completed for somewhat later publication the calculation of tables giving to fifteen decimal places the natural values of the sines, tangents, and secants of angles up to 90 degrees, in steps of ten seconds.

At Columbia University leaves of absence have been granted to Professor T. S. Fiske for the first half year and to Professor H. B. Mitchell for the second half year of 1914–1915.

At a recent meeting of the New York section of the Association of the Teachers of Mathematics of the Middle States and Maryland, addresses were delivered by Professor C. J. Keyser on "The human worth of rigorous thinking" and by Professor H. E. Hawkes on "How can we teach our pupils in geometry to think?"

The Fifty-Second Annual Convention of the National Education Association will be held in St. Paul, Minn., July 4–11, 1914. The meetings of the National Council will begin on Saturday, July 4. Educational Sunday will be observed on July 5, and the general sessions will open on July 6.

In *L'Enseignement Mathematique* for January, 1914, N. Gennimatas proves that a Heronian triangle, that is a triangle such that the area is a rational function of the sides, is always similar to a triangle whose sides may be written in the form: $a = x^2 + y^2$, $b = (1 + y^2)x$, $c = (1 + x)(y^2 - x)$, the area of the latter being equal to cxy.

"Memorabilia Mathematica or the Philomath's Quotation Book" is a volume by Dr. R. E. Moritz of the University of Washington, just published by the Macmillan Company. The following topics are discussed in the twenty-one chapters: definitions and object of mathematics, the nature of mathematics, estimates of mathematics, the value of mathematics, the teaching of mathematics, study and research in mathematics, modern mathematics, the mathematician,

mathematics as a fine art, mathematics as a language, mathematics and logic, arithmetic, algebra, geometry, the calculus and allied topics, the fundamental concepts of time and space, paradoxes and curiosities.

In *Science* for April 24, 1914, Professor C. N. Moore, of the University of Cincinnati, calls attention to an address by Professor Thorndike, of Columbia University, before the Cincinnati Schoolmasters' Club in which some sweeping statements in regard to the educational value of mathematics are made. As against these statements, Professor Moore recalls the article of Professor Keyser, head of the department of mathematics at Columbia University, published in *Science* for December 5, 1913, on "The human worth of rigorous thinking." It is well worth while for all friends of mathematics to consider carefully the claims set forth by both of these writers.

On April 14, 1914, there was celebrated at the University of Palermo the thirtieth anniversary of the foundation of the Circolo Matematico of Palermo. On that occasion a gold medal was presented to G. B. Guccia, the founder of the society and of its organ, the *Rendiconti*. As the society is international, the co-operation of mathematicians of all countries was invited. The local committee of arrangements consisted of Professors M. Albeggiani, F. Raffaele, G. Bagnera, G. Pitrè, A. Venturi, M. Gebbia and M. de Franchis.

The interest taken in Italy in the four hundredth anniversary of the birth of Tartaglia is made evident by the notices appearing in the daily press. *La Tribuna* in Rome printed on February 9 a long notice of Tartaglia, written by the engineer V. Tonni-Bazza, who has made a special study of the life and works of the sixteenth century algebraist. Antonio Favaro of the University of Padua has published several articles about Tartaglia in technical journals.

The February issue of *Isis*, the new journal on the history of science, edited by George Sarton and published in Belgium, completes the first volume. It is a substantial volume of 824 pages. The last number contains the following articles on mathematics: "Les tendances actuelles de l'histoire des mathématiques" by the editor, "Le glorie matematiche della Granbretagna" by Gino Loria, "The origin of Cauchy's conceptions of a definite integral and the continuity of a function" by Philip E. B. Jourdain. The international character of the journal is indicated by the fact that the three contributors just named have written in French, Italian and English. A fourth original article is in German, by Ernst Bloch, on "Die chemischen Theorien bei Descartes und den Cartesianern."

The twenty-fifth anniversary of the founding of Teachers College, Columbia University, was celebrated February 21. At the end of the present academic year, the college will place its work, with but one exception, on a graduate basis. According to the *Columbia University Quarterly*, "the normal schools supply the rank and file of teachers of the common schools; the state universities, through their departments of education, are equipping the rank and file of the teachers of the secondary schools. There remains however the preparation of the super-

visors and administrators in these fields and the equipping of the teaching staffs of all these professional faculties. It is to these fields, then, that Teachers College prepares to confine itself . . . ."

Beginning with January 1, 1914, the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht* is being edited by H. Schotten, W. Lietzmann and E. Grimsehl. Heretofore it was edited by H. Schotten alone. B. G. Teubner continues to publish this journal. In the third number of Vol. 45, W. Lietzmann makes a report on the teaching of the differential and integral calculus in oberrealschulen, realgymnasien and gymnasien at the present time. The calculus is taught in most of these schools, but is developed with less rigor than in university courses. The use of the derivative predominates over that of the differential. There is little fusion of the differential and integral calculus, the latter being taken up only in part or neglected altogether. The introduction of the calculus into these schools is generally looked upon as a step forward, in spite of the lack of coördination of the courses with the university courses. As Klein says, the present is a period of transition. One of the vital questions is the degree of rigor with which the subject should be developed in these schools.

The earlier and wider introduction of the calculus receives attention also in England. In the December, 1913, number of the *Mathematical Gazette*, C. S. Jackson gives the first instalment of a paper, entitled "The Calculus as an item in school mathematics," which he read before the London branch of the Mathematical Association. Twenty-five years ago only a few of the best students took up calculus; skill in manipulation was overemphasized, applications were neglected. More recent demands made by physicists and engineers have led to a cleavage in the modes of initiating boys into the calculus. "The rigorist puts minute logical precision first—the practical man asks us to dwell on the utility of the calculus. Of the two the rigorist, in relation to the schoolboy, is by far the more astray. . . . We as teachers are just learning that you cannot teach people to generalize by throwing ready-made generalizations at them—that a grip on the concrete facts must precede a critical analysis of them—that there is no worse mistake than to jump before you come to the fence." Jackson's aim is expressed by himself as follows: "It is the object of this paper to argue that in broad outline there is one way—one fairly definite sequence—to which the teaching of the subject ought with variations to conform." In the part of his paper thus far published he enters upon a description of the preliminary work.

The following courses in mathematics will be offered at the Summer session of the University of Chicago, June 15 to August 28, 1914: By Professor G. A. Bliss, plane analytic geometry, and metric differential geometry; by Professor D. R. Curtiss, differential calculus, and functions of a complex variable; by Professor L. E. Dickson, theory of equations, theory of invariants (1st term), and linear algebra (2d term); by Professor Kurt Laves, analytic mechanics, and observatory practice; by Professor A. C. Lunn, geometrical optics, and electron theory of electromagnetism; by Professor C. N. Moore, college algebra,

and integral calculus; by Professor E. H. MOORE, solid analytic geometry (1st term), and integral equations in general analysis (1st term); by Professor F. R. MOULTON, problem of three bodies; by Professor G. W. MYERS, teaching of elementary school mathematics, and teaching of secondary school mathematics; by Professor H. E. SLAUGHT, plane trigonometry, and elliptic integrals; by Professor J. W. A. YOUNG, college geometry, and critical review of secondary mathematics. Also private reading and research in both pure and applied mathematics is conducted for advanced students. The mathematics club, consisting of all instructors and advanced students in mathematics and mathematical astronomy, meets weekly for discussion of research and pedagogic questions.

At the recent meeting of the Ohio Association of Mathematics and Science Teachers, a number of teachers of mathematics discussed the advisability of forming a mathematics section under the Ohio Academy of Sciences. Another suggestion was that the group of teachers concerned petition the council of the American Mathematical Society, of which many of them are members, for permission to organize a Junior Section of the Society for the purpose of conducting meetings at which papers of less formal character and of more direct bearing on mathematical teaching than is usual at regular meetings of the Society could be presented and discussed. Such meetings would be of value both to those who find it too expensive and inconvenient to attend regular meetings of the Society as now constituted, and to those whose interests lie chiefly in the field of mathematical teaching. It was further suggested that notices and reports might be published in the AMERICAN MATHEMATICAL MONTHLY, and that this journal might be made an official organ of such junior sections in case the plan were eventually recognized and approved by the Society.

Erratum.—In Professor Roever's article in the December, 1913, issue, the last sentence at the bottom of page 303 should be modified to read as follows: "This is an element in which a plane through the vertex and perpendicular to the internal bisector of the angle P_1PP_2 (Fig. 3) cuts the conical roof."

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THE CONSTRUCTION OF CONICS UNDER GIVEN CONDITIONS.

By B. M. WOODS, University of California.

The following discussion treats by synthetic methods the matter presented by Professor M. W. Haskell[1] in a paper of the same title, where the treatment is for the most part analytic.

SECTION I. CONICS DEFINED BY FIVE POINTS OR BY FIVE TANGENTS.

Theorem I. *The eight vertices of two quadrangles with the same diagonal triangle lie on a conic.*

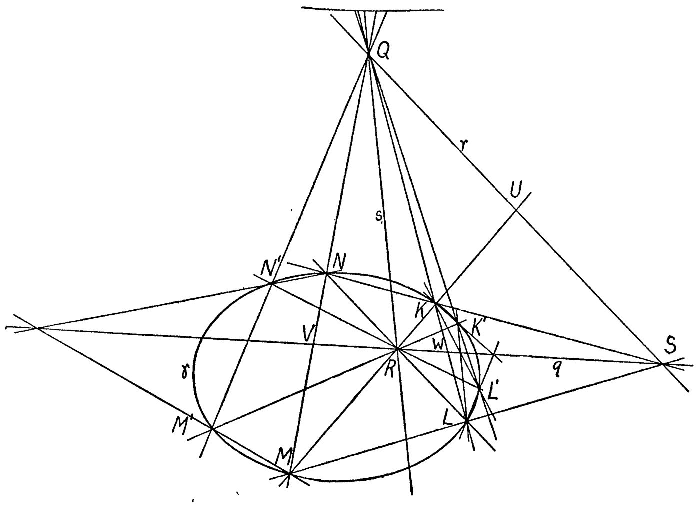


Fig. 1.

Consider a point Q (Fig. 1) and its polar q with respect to a given conic γ. If a point R is selected on q, its polar r passes through Q and cuts q in a point S, whose polar s is the line QR. A self-polar triangle QRS is thus determined.

[1] *Bulletin of the Amer. Math. Soc.*, 2d Series, Vol. XI, No. 5, pp. 268–273.

If now an arbitrary ray QK be drawn through Q, cutting the conic in K and L, and the rays KR and LR be drawn intersecting the conic in M and N respectively, the ray QN will pass through M. For, by the fundamental properties of the polar, Q, K, W, L are four harmonic points.[1] QN cuts the four harmonic rays QR, KR, WR, and LR in four harmonic points. Also the fourth harmonic of N with respect to Q and V must be on the conic. Hence, QN meets KR on the conic. Similarly NK and ML may be shown to intersect at S. Hence, an arbitrary ray through Q determines uniquely a quadrangle $KLMN$ with its vertices on the conic γ and with the diagonal triangle QRS.

Now a quadrangle is uniquely determined by its diagonal triangle and one vertex as follows: In Fig. 2, let QRS be the diagonal triangle and M the given vertex. Join M to Q, R, and S. Construct U, the harmonic conjugate of V (the intersection point of MS and QR) with respect to Q and R. Join U to S, determining N and K. Draw NR and KQ. This construction gives the quadrangle $KLMN$ desired, for, from the properties of the quadrangle, the two sides passing through S cut QR in points that are harmonic conjugates of Q and R.

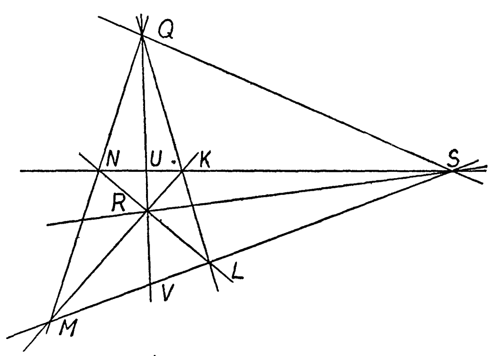


Fig. 2.

If, then, the diagonal triangle QRS is given, one vertex K determines the quadrangle $KLMN$. If another vertex K' be assigned arbitrarily, it determines with K, L, M, N a conic γ. It also determines a quadrangle $K'L'M'N'$ with diagonal triangle QRS and vertices on the conic γ. Since these two quadrangles are chosen arbitrarily among those that have the diagonal triangle QRS, the theorem is established.

From this follows a simple construction by points of a conic of which five points are given.

Construction: Construct the complete quadrangle of any four of the points with its diagonal triangle. The fifth point and the diagonal triangle obtained determine a second quadrangle whose vertices lie on the required conic. By taking the quadrangle of each four points in turn, we locate fifteen new points in this manner. Combining these in various ways, we may locate as many points as desired.

[1] See Reye, "Geometrie der Lage," Chapter VIII.

We shall omit here and in general in what follows, all theorems obtained by dualization.

Theorem II. *The lines joining pairs of vertices of two quadrangles with the same diagonal triangle form a quadrilateral with this diagonal triangle.*

Let us consider Fig. 1 again. QR was chosen originally as an arbitrary ray through Q. Hence, the demonstration of Theorem I includes the proof that, if QK and QK' be arbitrary rays through Q, KK' and LL' will meet on q, the polar of Q. So, also, will $K'L$ and KL'. Hence, in the quadrilateral formed by KK', LL', MM', and NN', we have KK' and LL' intersecting on q and similarly MM' and NN'. Likewise, we may show that KK' and NN', and LL' and MM' intersect on s, and finally KK' and MM', and LL' and NN' on r. Hence, the diagonal triangle of the quadrilateral is QRS, the same as that of either quadrangle $KLMN$ or $K'L'M'N'$.

If now K' approaches K along the conic, L', M', and N' approach L, M, and N respectively and the sides of the quadrilateral approach as limiting positions, the tangents to the conic at the vertices of the quadrangle. From which follows

Theorem III. *The tangents at any four points of a conic form a quadrilateral which has the same diagonal triangle as the quadrangle of those four points.*

By combining this theorem with the dual of Theorem I, viz., the eight sides of two quadrilaterals with the same diagonal triangle envelop a conic, we obtain the following

Theorem IV. *The eight tangents at the points of intersection of two conics have the same diagonal triangle as the quadrilateral of the four tangents to either one of the conics, at these points. These two quadrilaterals have the same diagonal triangle and, consequently, their sides envelop a conic.*

The second part of the theorem is the dual of the first. The conic mentioned in the second part is the covariant conic[1] of the two given conics, and is the locus of points from which the pairs of tangents to the given conics are harmonic conjugates of each other. A synthetic discussion has been given by Professor Lehmer.[2]

Theorem V. *The quadrilateral of the four common tangents to two conics has the same diagonal triangle as the quadrangle of the intersection points of the two conics.*

For let PP' be one of these common tangents, P and P' being the respective points of contact. Then if Q, R, S; Q', R', S', respectively, be the remaining vertices of the quadrangles determined by P and P', these quadrangles having the same diagonal triangle as the quadrangle of the four points of intersection of the two conics, then QQ', RR', and SS' will be the remaining sides of the quadrilateral determined by PP', having the same diagonal triangle as the quadrangles just determined. Q, R, S will lie on the first conic; Q', R', S' on the second; and QQ', RR', SS' will be tangent to both.

[1] See Salmon, *Conic Sections*, pp. 306 and 344.

[2] *Amer. Math. Monthly*, Feb., 1908, "A Discussion by Synthetic Methods of the Covariant Conic of two Given Conics."

Section 2. Involutory Quadratic Transformation. Conics Through Four Given Points and Tangent to a Given Line.

An involution of rays may be regarded as composed of two superposed projective pencils of rays where the correspondence is such that a given ray has but one corresponding ray whether it be considered as belonging to one or the other of the superposed pencils. Consider two such involutions of rays with centers at A and B (Fig. 3), to be called the involutions A and B. To a point Q of the plane corresponds a point Q' obtained by finding the point of intersection of AQ' and BQ', the rays at A and B corresponding to AQ and BQ respectively. If a point Q moves along a ray q, it projects to A and B in two perspective pencils

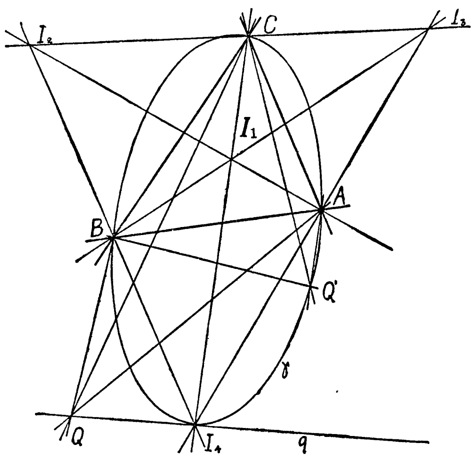


Fig. 3.

of rays. The corresponding rays AQ' and BQ' in the involutions A and B describe pencils of rays projective to the pencils AQ and BQ respectively and hence projective to each other. Q', therefore, describes a conic through A and B. It has been noted that to every point Q of the plane there is a corresponding point Q', and conversely. There are, however, special points and lines for which the correspondence is not unique. Any point of the line AB, for instance, corresponds to the same point C, the point of intersection of the rays at A and B corresponding to AB considered first as a ray of the involution A and then as one of the involution B. Conversely, AB may be considered as corresponding to C.

Denote now by γ the conic described by Q', as Q describes q. As Q' describes γ, the pencils of rays CQ' and CQ are projective to each other. For the pencil CQ' is projective to the pencil AQ' and the pencil CQ is perspective to the pencil AQ, the pencils AQ and AQ' are projective to each other by hypothesis. Moreover, Q and Q' are related to each other reciprocally by means of the involutions A and B. Hence, the pencils CQ' and CQ, superposed at C, give an involution of rays at that point which may be paired with the involution A or B to give the original transformation. Suppose that A and B, which were chosen arbitrarily,

have double rays. The intersection point of a double ray of A with a double ray of B is a self-corresponding point in this transformation of the plane. Hence, it must also lie on a double ray of C. Therefore, the involution C has double rays, and two of the intersections of the double rays of A and B lie on one of them, while the other two lie on the other. Hence, the six double rays of A, B, and C intersect in four points and, consequently, form a quadrangle whose diagonal triangle is ABC. Denote the four intersection points by I_1, I_2, I_3, I_4. These are obviously the only self-corresponding points of the plane. It should also be noted that since the involution C has been added, all the vertices of the triangle ABC correspond to its opposite sides, and conversely.[1]

Now, if with the involutions A and B arbitrarily chosen and the involution at C determined as above, we consider the point Q moving on an arbitrary ray q, its corresponding point describes a conic γ, through A, B, and C. If, as Q describes q, Q' crosses q, Q and Q' are at that instant the points of intersection of γ and q. Hence the

Theorem VI. *The conic into which a line is converted by the involutory quadratic transformation meets this line in a pair of corresponding points, the only pair of such points on the line. If the line is tangent to its corresponding conic, the point of tangency is a self-corresponding point of the transformation.*

Corollary. The conic corresponding to a line through a self-corresponding point is the conic through A, B, C which is tangent to the given line at the self-corresponding point. The conic of Fig. 3 is drawn in this position for convenience.

Moreover, any two corresponding rays of an involution are harmonic conjugates of the double rays. Hence, the rays joining Q and Q' to A, B, and C are harmonic conjugates of the double rays at those points. Since the pencil of conics through the self-corresponding points I_1, I_2, I_3, I_4 cuts out an involution of points on any ray of the plane, in particular on q, the double points of this involution mark the points of tangency of the two conics of the pencil tangent to the line. The three pairs of opposite sides of the quadrangle of the self-corresponding points are the pairs of double rays of the three involutions A, B, and C; and they cut q in three pairs of points of the involution. Now, since Q and Q' on q are harmonic conjugates of each pair of these points, they are the double points sought for. Whence, we write

Theorem VII. *The line joining a pair of corresponding points, Q and Q', in an involutory quadratic tranformation of the plane, is tangent to two conics of the pencil of conics through the four self-corresponding points of the transformation, the points of tangency being Q and Q'.*

Conversely, *if there are two conics of this pencil of conics tangent to an arbitrary line of the plane, the points of tangency are corresponding points in the transformation.*

The reality of the conics of the pencil through I_1, I_2, I_3, I_4, tangent to a given line, depends on whether the involution of points on that line determined by the

[1] For a more complete discussion of this transformation see Professor Lehmer's article in AMER. MATH. MONTHLY, "On the Combination of Involutions," March, 1911.

pencil of conics has double points or not. The intersections of the line with the opposite sides of the quadrangle give pairs of corresponding points on the line. These pairs of points must not divide each other if the involution is to have double points. Therefore, the given line must cut each pair of opposite sides of the quadrangle of the four points either in the segments upon which a vertex of the diagonal triangle lies, or else in those segments upon which it does not lie. If this condition is fulfilled for one pair of sides, it is fulfilled for the remaining pairs.

In this transformation, the points of intersection of the given line with the sides of the quadrangle of the self-corresponding points are points on the double rays of A, B, and C. A point on one of these double rays is evidently converted into a point on that same ray,—its harmonic conjugate with respect to the two self-corresponding points on the ray.

Theorem VIII. *The conic into which a given line is converted by the involutory quadratic transformation passes through the harmonic conjugates of the intersections of that line with the six sides of the quadrangle whose vertices are* I_1, I_2, I_3, I_4.

The conic corresponding to the line at infinity is then the conic passing through the middle points of the sides of the quadrangle and through the vertices of its diagonal triangle.

It is to be noted that a pair of corresponding points, Q and Q', together with the vertices of the diagonal triangle, completely determine the quadratic transformation, as follows: Q and Q' must project to A (Fig. 3), for example, in a pair of corresponding rays. Likewise, the rays AB and AC are corresponding. This determines the involution at A. Similarly, at B and C. The self-corresponding points may be found as intersections of the double rays of the several involutions.

Section 3. Conics with Three Tangents and Two Points Given.

The proof for the number of conics with three points and two tangents given has been established by Steiner.[1] The following proof is carried out more or less along the same lines. The problem at hand is the number and construction of the conics fulfilling the given conditions, the solution of which depends on an auxiliary theorem, which will be discussed first.

Theorem IX. *The range of conics tangent to four lines determines at any point of the plane an involution of rays composed of the pairs of tangents from the points to the conics of the range. If the four fixed tangents should coincide in pairs so that the range of conics consists of those tangent to two fixed lines with fixed points of contact, the involution at any point of the plane has double rays.*

For, in particular, the line joining this point to the point of intersection of the two fixed lines is a double ray of the involution. This follows from the fact that the three pairs of opposite vertices of the complete quadrilateral of the four fixed lines, being the degenerate conics of the range, project to any point S in three pairs of corresponding rays of the involution determined at S. In our case, two pairs of opposite vertices have fallen together in one point, the point of intersection of the fixed rays, which therefore projects as a double ray.

[1] *Vorlesungen über Synthetische Geometrie*, Zweiter Teil, p. 236, ed. 1876.

Apply this to the solution of the problem. Call the given lines a, b, and c; the opposite vertices of the triangle they determine A, B, and C; and the given points L and L' (Fig. 4). An involution of rays is determined at A, for instance, by the rays b and c as one pair and LA and $L'A$ as the other. Similarly, at B and C. Call the double rays of the involution at A, g and h; of the involution at B, g' and h'; of the involution at C, g'' and h''. Since this method of establishing the involutions determines an involutory quadratic transformation of the plane which has L and L' for corresponding points and has the centers of its involutions at A, B, and C, the intersection points of the double rays at A, B, and C will be

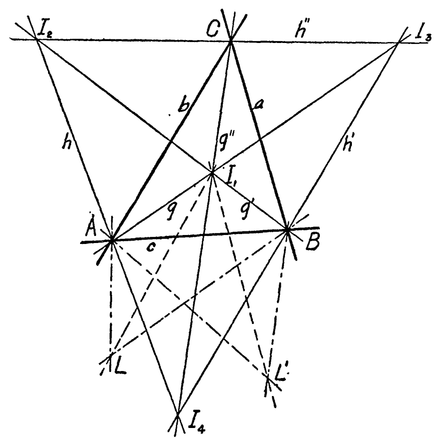


FIG. 4.

the four vertices of a complete quadrangle with diagonal triangle ABC. Call these intersection points I_1, I_2, I_3, I_4. They are naturally the self-corresponding points of the transformation. I_1 is the intersection point of double rays of A, B, and C; say, g, g', and g''. The conics tangent to I_1L and I_1L' at L and L', respectively, will determine an involution of rays at A, as at any point of the plane, made up of the tangents from A to the respective conics. g is a double ray of this involution, and AL and AL' are corresponding rays. For g joins A to the point of intersection of the fixed tangents to the range of conics, and L and L' constitute one of the degenerate conics of the range. But this is the involution previously determined at A. Likewise, at B and C. Now, in this involution at A, b and c are corresponding rays; at B, a and c are corresponding rays. Hence, the particular conic of this range of those tangent to I_1L at L and I_1L' at L' which is itself tangent to c, is also tangent to a and b. Since it also passes through L and L', it is one of the conics required. Each of the four self-corresponding points gives one such conic; viz., the conic tangent to the rays joining it to L and L' at L and L', and tangent to the three rays a, b, and c. In each case, LL' is the polar of the self-corresponding point joined to L and L'.

Hence, the following theorems, which are grouped for convenience.

Theorem X. *There are four conics tangent to three given rays and passing*

through two given points. The ray joining the two given points being a common chord, the conics are determined as soon as its poles are known. Its four poles are the self-corresponding points of the involutory quadratic transformation of which the given lines constitute the fundamental triangle, and by which the given points are interchanged.

These conics will all be real if the involutions determined at A, B, and C

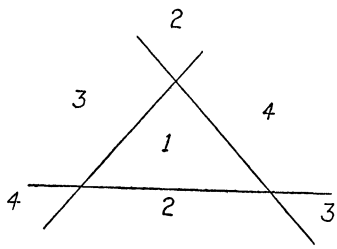


FIG. 5.

have double rays. For this to happen, the rays joining A, for instance, to L and L' must not be divided by b and c. Hence, all the conics will be real if L and L' lie together in one of the four sections into which the given lines divide the plane[1] (Fig. 5). Otherwise, they are all imaginary, since two of the involutions have no double rays.

A SIMPLE METHOD OF CONSTRUCTING THE NORMALS TO A PARABOLA.

By S. G. BARTON, University of Pennsylvania.

We know by Harvey's Theorem that the feet of the three normals to a parabola from any point lie on a circle which passes through the vertex. The converse is also true, the normals whose feet lie on a circle through the vertex are concurrent. For let (d, e) be the center of such a circle. Its equation then is $x^2 + y^2 - 2dx - 2ey = 0$, which intersects the parabola $y^2 = 4px$ in points whose ordinates are given by

$$y^3 - 8p(d - 2p)y - 32p^2 e = 0.$$

Since this lacks the second term the sum of its roots is zero which is the condition for concurrent normals.

The equation giving the ordinates of the feet of the normals through a point (h, k) is

$$y^3 - 4p(h - 2p)y - 8p^2 k = 0.$$

But these equations must be equivalent, hence by equating coefficients $2d - 4p = h - 2p$ and $32e = 8k$; therefore $d = \frac{1}{2}h + p$ and $e = \frac{1}{4}k$.

[1] See Annie Dale Biddle, "Constructive Theory of the Unicursal Plane Quartic by Synthetic Methods," Univ. Calif. Publ. Math., Vol. 1, No. 2.

Hence to construct the normals from the point (h, k), find the feet by constructing a circle whose center is $(\frac{1}{2}h + p, \frac{1}{4}k)$ and radius such that it passes through the vertex of the parabola. The intersections of the circle with the parabola other than the vertex are the feet of the three normals. If the point (h, k) is on the convex side of the evolute of the parabola, the three normals are real and distinct; if on the evolute two are coincident, all three coinciding at the cusp; if on the concave side, two normals are imaginary. Under the same conditions the center of the circle will bear the same relation to the curve $(x - 2p)^3 = \frac{27}{2}py^2$.

If there is difficulty in locating the exact point of intersection, it will often be of aid to recall that the sum of the three ordinates must vanish.

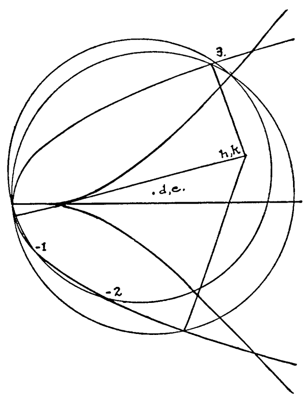
A graphical solution for cubic equations may be based upon the preceding construction for the normals of a parabola, since we have solved the cubic which gives the ordinates of the feet of the normals, namely,

$$y^3 - 8p(d - 2p)y - 32p^2 e = 0.$$

Hence in this manner we can solve any cubic. Let the cubic (which must be transformed so as to lack the second term), be of the form

$$y^3 - ay - b = 0.$$

Equating coefficients we get $d = a/8p + 2p$, $e = b/32p^2$. Hence construct any parabola and describe a circle whose center is (d, e), as determined above, and radius such that it passes through the vertex. Then the ordinates of the inter-



sections, other than the vertex, are the roots of the equation. One parabola only need be drawn, but for convenience the unit of measure should be changed to suit the coefficients. It will usually be found desirable to choose $p = 0.2a$ approximately.

For example to solve the equation $x^3 - 7x - 6 = 0$, let p be 0.5, whence $d = 2.75$ and $e = 0.75$.

This center and the intersections are shown on the figure. Whence we read the roots -1, -2 and 3. If the equation had been $x^3 - 700x - 6{,}000 = 0$, that is, the roots ten times as large, we would take $p = 5$ and the figure would be identical, the unit only being changed. The curve $(x - 2p)^3 = \frac{27}{2}py^2$, which gives the criterion for the nature of the roots is also shown. The roots are: one real and two imaginary, two equal or three real and unequal, according as the center (d, e) is to the left of, on, or to the right of, this curve. The accuracy of the solution varies with the nature of the intersections.

SOME PROPERTIES OF THE NORMALS TO PARABOLAS.

By S. G. BARTON, University of Pennsylvania.

There are in general three normals to a parabola from any point. If the point lies on the evolute of the parabola, two of the normals coincide (all three coincide at the cusp); if on the concave side of the evolute two normals are imaginary; if on the convex side, the three normals are real and distinct. The following properties referring chiefly to normals drawn from points on the evolute were found in an investigation for another purpose.

Consider the parabola $y^2 = 4px$, whose evolute is $27py^2 = 4(x - 2p)^3$. If (a, b) is a point on the evolute, then

(1) The equations of the coincident and single normals are respectively

$$y - b = \sqrt{\frac{a - 2p}{3p}}\,(x - a) \qquad \text{and} \qquad y - b = -2\sqrt{\frac{a - 2p}{3p}}\,(x - a)$$

and coördinates of the feet of these normals are

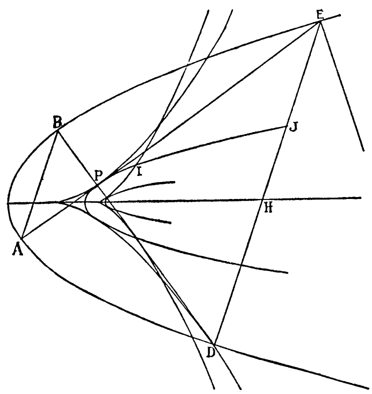
$$\left(\frac{a - 2p}{3},\ -2\sqrt{\frac{(a - 2p)p}{3}}\right) \qquad \text{and} \qquad \left(\tfrac{4}{3}(a - 2p),\ 4\sqrt{\frac{(a - 2p)p}{3}}\right).$$

Thus the abscissa of the foot of the single normal is four times that of the double normal, and the ordinate of the foot of the former is minus twice that of the latter. The abscissa of the foot of the double normal is one third of the difference between the abscissæ of the point (a, b) and the cusp of the evolute, and the difference of the abscissæ of the feet is equal to the difference for the point and cusp.

(2) The x intercepts are $\frac{1}{2}(a + 4p)$ and $\frac{1}{3}(4a - 2p)$. Hence the foot of the ordinate of the point (a, b) trisects the distance between the intersections of the normals with the x axis, being nearer the single normal. The difference of the two intercepts is $(a - 2p)$ which is the distance from the foot of the ordinate to the cusp. Thus the single normal cuts the x axis four times as far from the cusp as the double normal. Similarly the foot of the abscissa trisects the distance between the intersections of the normals with the y axis, being nearer the double normal. The x intercept of the double normal is a mean proportional between the abscissæ of the points in which it intersects the parabola.

(3) The difference of the squares of the lengths of the normals (point to curve) equals three times the square of the difference of the abscissæ of the point and the cusp. The square on the line joining the feet of the normals is three times the square on the line joining the vertex with the foot of the double normal. The double normal (point to curve) is divided by the axis in a ratio which is twice that into which the axis divides the single normal. The locus of the intersection of two normals which are perpendicular to each other is the parabola $y^2 = p(x - 3p)$. This locus touches the evolute at $\left(\frac{1}{2}p, \pm \frac{\sqrt{2}}{2}p\right)$.

(4) These points and the normals through them are particularly interesting. The single normal through the upper of these points, P in the figure, is $y = -\sqrt{2}(x - 4p)$. This line is normal to both parabolas, normal to the evolute at P and tangent to the evolute at D.



If we draw the evolute of the parabola of perpendicular normals, $y^2 = p(x - 3p)$, which we call the second parabola, and its parabola of perpendicular normals, and so on we get a series of parabolas and evolutes each having the line $y = -\sqrt{2}(x - 4p)$ as a normal. The latus rectum of each parabola is one fourth that of the preceding and the cusp of each evolute is at the foot of the ordinate of the point in which the common normal cuts the parabola. DB is the radius of curvature at B.

(5) The similar tangents and normals to the parabolas and evolutes at the points where the common normal cuts them are parallel. The tangent at the lower point of intersection in each case passes through the focus of the preceding parabola.

(6) The common normal and its correspondent constitute the envelope of the evolutes of the series of parabolas $y^2 = 4p'(x - 4p + 4p')$, in which p' is the variable parameter. The parabolas considered are particular ones of the series.

(7) The length of the common normal is $6\sqrt{3}\,p$. It is trisected by the axis,

bisected at one point of intersection with the second parabola, and quadrisected by the other, smaller divisions being made by the other parabolas. The length of the double normal is $6\sqrt{6}\,p$. It is divided by the axis into parts which are as 1 to 5, and by the evolute into parts which are as 5 to 27 and 1 to 3.

(8) The line AB, joining the feet of the normals, passes through the focus (as is true for any pair of perpendicular normals), is $4\frac{1}{2}$ times p in length, is trisected at the focus, is parallel to DE (joining the other ends of the normals through P) and equal to $\frac{1}{3}DE$. Both AB and DE are bisected by the diameter through P and are of course parallel to the tangent at its extremity.

(9) DE is itself a normal to the parabola at D. If we draw the normal at E, then DE bisects the angle between the other two normals through E, the two to the right forming with the axis an isosceles triangle.

(10) The cusp and the points P and I are collinear, P being the point of bisection. The focus and the points P and J are collinear, P being the point of quadrisection, and the join being perpendicular to the normal at E.

(11) P is equidistant from B and the focus. The vertex, A, B and the middle point of BD lie on a circle whose center is on AE. The point I is equidistant from the cusp and B. The focus is equidistant from the intersections of the evolute with the parabola and H. A circle with BD as a diameter passes through the vertex. D and B are equidistant from J and $AB = JE$.

(12) The perpendicular bisector of DH is tangent to the second parabola where BD cuts it and the normal to the evolute at this point passes through J. The perpendicular to DE at H passes through B.

(13) The chord joining the vertex with the upper intersection of the parabola and evolute is perpendicular to BD and trisected by it.

(14) The third normal from any point on the second parabola is trisected by the axis.

(15) The areas between the normal chords AE and BD and the parabola are equal. Moreover the three areas between the chords BE, AB, and AD and the parabola are equal.

(16) From any point whatever (a, b) the sum of the X intercepts is $2(a+p)$, that is, twice the distance of the point from the directrix, and is constant for all points on a line perpendicular to the axis. Likewise the sum of the focal distances of the feet of the three normals is constant and equal to $(2a - p)$.

A SIMPLE ALGEBRAIC PARADOX.

By J. L. COOLIDGE, Harvard Universtiy.

Given two linear homogeneous complex equations

$$(a + bi)(p + qi) + (c + di)(r + si) = 0,$$
$$(a' + b'i)(p + qi) + (c' + d'i)(r + si) = 0. \tag{1}$$

In order that these should be compatible it is necessary and sufficient that

$$\begin{vmatrix} (a + bi) & (c + di) \\ (a' + b'i) & (c' + d'i) \end{vmatrix} = 0.$$

This complex equation is equivalent to the *two* real equations

$$(ac' - a'c) = (bd' - b'd),$$

$$(ad' + bc') = (a'd + b'c).$$

Both of these must be fulfilled if (1) is to subsist.

On the other hand equations (1) are equivalent to

$$\begin{aligned} ap - bq + cr - ds &= 0, \\ bp + aq + dr + cs &= 0, \\ a'p - b'q + c'r - d's &= 0, \\ b'p + a'q + d'r + c's &= 0. \end{aligned}$$

These are compatible when, and only when, a *single* equation is satisfied, namely

$$\begin{vmatrix} a & -b & c & -d \\ b & a & d & c \\ a' & -b' & c' & -d' \\ b' & a' & d' & c' \end{vmatrix} = 0.$$

Which answer is right?

ON A PURELY PROJECTIVE BASIS FOR THE THEORY OF INVOLUTION.

By D. N. LEHMER.

The theory of involution is usually based on foundations which are more or less metrical. It is customary to derive the more important properties by means of circles, or else by the use of the theory of the anharmonic ratio, which itself is usually based on metrical notions of areas and trigonometric functions. These methods have their advantages, but to the purist it seems unfortunate that the general, projective properties of involution should not be derived in a non-metrical way, and that the metrical properties should not be obtained, as is usual, by the introduction of the elements at infinity.

A purely projective discussion of the subject may be based on the following well-known and easily derived theorem on the complete quadrangle:[1]

[1] See Reye, *Geometry of Position*, Holgate's translation, page 36.

If two complete quadrangles, KLMN and K′L′M′N′, are so related that the homologous sides KL and K′L′ meet in a point A and the opposite pair, MN and M′N′, in A′; LM and L′M′ in B, and KN and K′N′ in B′; LN and L′N′ in C, while A, A′, B, B′ and C all lie on the same straight line u, then the sixth pair of sides, KM and K′M′, will also meet on that line.

The proof follows easily from Desargues's theorem and its converse. For consider the two triangles *KLN* and *K′L′N′* (Fig. 1): corresponding sides meet on *u;* therefore the lines *K′K, L′L,* and *N′N* all meet in a point *S*. But by the triangles *LMN* and *L′M′N′* we observe that *MM′* must also pass through *S*. Then since the lines joining corresponding vertices of the triangles *KMN* and *K′M′N′* are concurrent in *S*, we see that the pairs of corresponding sides meet in three collinear points *A′, B′, C′* which must all lie on *u*.

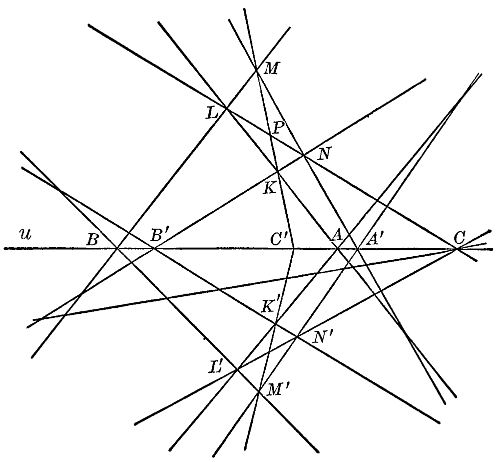


Fig. 1.

In this cut *A, A′*, and *B, B′* are points on line *u* and *KLMN* and *K′L′M′N′* are complete quadrangles constructed so as to have pairs of opposite sides passing through these points.

From the theorem it is seen that if *AA′*, *BB′* and *C* are given then *C′* is uniquely determined, and that independently of the particular quadrangle employed to construct it. It is clear also that three pairs of points are not generally so placed on a line that the three pairs of opposite sides of a complete quadrangle may be drawn through them. As a definition, then:

If three pairs of points on a line are such that the three pairs of opposite sides of a complete quadrangle may pass through them, the points are said to be three pairs in involution. Any two pairs being given, as many other pairs may be constructed as desired, for choosing any point *C*, the other point that pairs with it, called its conjugate, may be found by the above linear construction. The solution, in detail, is as follows: Through *A* and *A′* draw any two lines. Cut them by any line through *C* in the points *L* and *N*. Join *L* to *B*, and *N* to *B′*, thus obtaining the points *M* and *K* on the lines through *A′* and *A* respectively. The line*MK*

determines C'. All the pairs CC' thus obtained are said to belong to the same involution and taken together are said to separate the line in involution.

As C traverses the line, C' also traverses the line and the two point-rows thus obtained are projective. For fix K and N, then CN cuts out on KA a point-row L, which projects from B to a point-row M on NA', which, in turn, projects from K to C'.

It follows from this last remark that NC and KC' furnish two projective pencils, and that their intersection, P, generates a conic. This conic meets the line u in two points F_1 and F_2, real or imaginary, which are the *double points* in the involution; that is, they coincide with their conjugate points. Therefore: *An involution has two double points, or none.*

B and B' being a pair in the involution, the rays NB and KB' are corresponding rays. Therefore NB is tangent to the conic at N, and KB is tangent to it at K. NK is then the polar of B with respect to the conic. B and B' are thus conjugate with respect to the conic, and are thus harmonically separated by the two double points. We have then the theorem:

Conjugate points in an involution are harmonic conjugates with respect to the double points.

It is to be noted that, if the complete quadrangle is drawn with two pairs of opposite sides passing through the double points, the figure reduces to that used in defining four harmonic points.

Metrical theory. We introduce now that point at infinity the conjugate of which we will call the *center* and designate by O. Since conjugate points are harmonically separated by the double points it follows that O is the middle point of F_1F_2.

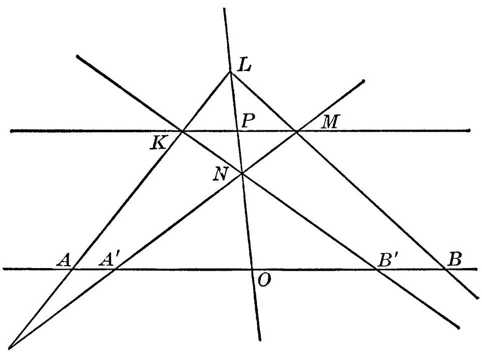


Fig. 2.

In constructing the center, the line MK (Fig. 2) is drawn parallel to u. It is then seen that the triangle MPL is similar to the triangle OLB, and the triangle PNK to the triangle ONB'; whence we may write

$$\frac{OB}{MP} = \frac{OL}{PL}$$

and

$$\frac{OB'}{PK} = \frac{ON}{PN};$$

whence

$$\frac{OB \cdot OB'}{MP \cdot PK} = \frac{OL \cdot ON}{PL \cdot PN}. \tag{1}$$

Also from the similar triangles OAL and PKL, combined with the similar triangles ONA' and MPN, we have

$$\frac{OA \cdot OA'}{MP \cdot PK} = \frac{OL \cdot ON}{PL \cdot PN}. \tag{2}$$

By combining (1) and (2) we have the fundamental relation

$$OA \cdot OA' = OB \cdot OB',$$

or in words:

The product of the distances from the center to a pair of conjugate points in the involution is constant.

From this theorem all of the usual developments may easily be made. The corresponding theory of lines in involution may be obtained in the same way, or by applying the principal of duality.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY.

The Pell Equation. By EDWARD EVERETT WHITFORD, Instructor in Mathematics in the College of the City of New York. Whitford, New York, 1912. iv + 193 pages. $1.00 postpaid.

Let A be a given non-square positive integer and consider the equation

$$x^2 - Ay^2 = 1 \tag{1}$$

from the point of view of determining the positive integers x, y which satisfy the equation. This is the so-called Pellian problem. Equation (1) is referred to as the Pell equation.

The principal purpose of the book under review is to give a history of the Pell equation (1) and of the more general equation $x^2 - Ay^2 = B$, where B is a given integer (positive or negative). There is added a table of the simplest solution of (1) for each value of A from 1,501 to 1,700 inclusive, similar tables having previously been given by other authors for the values of A less than 1,501. The book contains also a bibliography of the Pell equation, with references to over 300 authors, and a table of continued fractions for $\sqrt{A}$.

Of the historical section, covering pages 1–101 of the text, we shall speak further.

The author begins with a discussion of Euler's error by which the name of Pell improperly came to be associated with equation (1). There follows an interesting discussion of the relation of the Pellian problem to that of approximating the square root of a non-square integer. An account is then given of the most ancient Hindu solutions (pp. 6–9), of the most ancient Greek solutions (pp. 9–13), of the work of Theon (pp. 13–15), Archimedes (pp. 15–21), Heron (pp. 21–22) and Diophantus (pp. 22–26). Emphasis is put upon the later work of the Hindus (pp. 26–39), and brief mention is made of that of the Arabs (pp. 39–41) and of the Europeans before the time of Fermat (pp. 41–46).

This history of the early and largely unsuccessful struggle of mathematicians with the difficulties of the Pellian problem is very instructive for those who are engaged now in the development and extension of science and knowledge.

The French mathematician Fermat (1601–1665) was the first to assert that equation (1), where A is any non-square integer, always has an unlimited number of solutions in integers. To prove this and to obtain a method for finding all of these solutions he proposed it as a challenge problem to the English mathematicians. From his remarks about it, it is clear that he considered it a very difficult problem.

This challenge problem of Fermat's was solved by Lord Brouncker, at least so far as giving means for finding the solutions is concerned. On account of this success, the English mathematician Wallace congratulates his fellow-countryman Brouncker that he has "preserved untarnished the fame which Englishmen have won in former times with Frenchmen and has shown that England's champions in wisdom are just as strong as those in war." Fermat himself never published his solution of the problem.

It was Euler who first recognized the deep importance of the Pell equation for the general solution of the indeterminate equation of the second degree. He left several memoirs dealing with this question.

It was Lagrange who first proved in a rigorous manner that (1) always has solutions in integers. Of the work of Lagrange, Legendre said that it "must be considered the most important step which has been made up to the present time in the indeterminate analysis."

Our author treats further the work of Gauss and Dirichlet and of the relation of the Pellian problem to several other disciplines, namely, the following: quadratic forms, circle division, elliptic functions and hyperbolic functions.

The book on the whole will repay perusal both by the working mathematician and by the amateur in the theory of numbers.

R. D. CARMICHAEL.

PROBLEMS AND SOLUTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

SPECIAL NOTICE. In proposing problems and in preparing solutions, contributors will please follow the form established by the MONTHLY, as indicated on the following pages.

In particular, a solution should be preceded by the number of the problem, the name and address of the proposer, the statement of the problem, and the name and address of the solver.

The solution should then be given with careful attention to legibility, accuracy, brevity without obscurity, paragraphing and spacing, having in mind the form in which it will appear on the printed page.

Please use paper of letter size, write on one side only, leaving ample margins, put one solution only on a single sheet and include only such matter as is intended for publication.

Drawings must be made *clearly* and *accurately* and an extra copy furnished on a *separate sheet* ready for the engraver.

Unless these directions are observed by contributors, solutions must be entirely rewritten by the committee or else rejected.

Selections for this department are made two months in advance of publication.

Please send all solutions direct to the chairman of the committee.

MANAGING EDITOR.

ALGEBRA.

416. Proposed by H. O. HANSON, East Elmhurst, N. J.

Find the nth term and the sum of n terms of the series obeying the relation $u_i = u_{i-1} + 2u_{i-2}$ in terms of n and the first two terms, u_1 and u_2, these two terms being arbitrary.

417. Proposed by ELMER SCHUYLER, Brooklyn, New York.

Solve $E(x^2) - E(3x) = 7$, where $E(m)$ is the largest integer in m. The value of x is to be in the form $4 + y/32$ where y is an integer.

GEOMETRY.

444. Proposed by S. A. COREY, Hiteman, Iowa.

Let $ABCDE$ be a pentagon, plane or gauche, with sides AB, BC, CD, DE, and EA. Bisect BC and DE in H and K respectively. Extend AB from B to B', and AE from E to E'. On AB' take sects AP and AV, and on AE' take sects AL and AT. Draw AD, AC, AH, AK, and DT. Let a, b, c, and d equal AL/AE, AT/AE, AV/AB, and AP/AB, respectively. Extend (or contract) AC from C to W, and AD from D to S, making $AW = a \times AC$ and $AS = d \times AD$. Draw LM and PN parallel to, and of the same currency as, AD and AC, respectively, and of lengths $c \times AD$ and $b \times AC$, respectively. Draw AM, AN, ST, and WV. Draw DQ and VX parallel to, and of the same currency as, CB and TS respectively. Prove that $2(ad + bc)(AK \times AH \times \cos KAH + KE \times HC \times \cos QDK) = AM \times AN \times \cos MAN + TS \times VW \times \cos WVX$.

Suggestion: Make use of the identity

$$[(w + x)a + (w - x)c][(z + y)d + (z - y)b] + [(w + x)b + (w - x)d][(z + y)c - (z - y)a] = 2(ad + bc)(wz + xy),$$

and solve with w, x, y, and z vectors, in a manner similar to that employed by the proposer in the solution of problems 377 and 383 in the May and October, 1911, numbers of the MONTHLY. The results in certain special cases are instructive, *e. g.*, when one or more of the sides, AC, CD, DE, are zero; when two or more of the sides lie in the same right line; when all the sides are tangent to a sphere, circle, ellipse, etc.; when the angles given in the above equation are multiples of some angle, etc.

445. Proposed by CLIFFORD N. MILLS, South Dakota State College.

Given the perimeter of a right triangle ABC and the perpendicular BD falling from the right angle B to the hypotenuse AC, to determine the sides of the triangle.

446. Proposed by S. G. BARTON, University of Pennsylvania.

Prove any one or more of the sixteen theorems, stated without proof, in the article in this issue of the MONTHLY (pages 182–184) on "Properties of the Normals to a Conic."

CALCULUS.

366. Proposed by I. A. BARNETT, University of Chicago.

Compute the definite integral $\int_a^b \sin^{-1} x dx$ where $0 \leqq a \leqq 1$ and $0 \leqq b \leqq 1$, by direct summation.

367. Proposed by C. N. SCHMALL, New York City.

Show that the volume enclosed by the surface $(x^2 + y^2 + z^2)^5 = (a^3x^2 + b^3y^2 + c^3z^2)^2$ is $\frac{4}{9}\pi(a^3 + b^3 + c^3)$.

MECHANICS.

295. Proposed by B. F. FINKEL, Drury College.

A homogeneous hollow cylinder, whose inner radius is half of its outer radius, rolls without slipping down a plane inclined at an angle α to the horizontal. Find its acceleration.

[From Prescott's *Mechanics of Particles and of Rigid Bodies.*]

NUMBER THEORY.

218. Proposed by ELIJAH SWIFT, Princeton, N. J.

If p is prime and > 3, show that $\sum_{a=1}^{a=p-1} 1/a^2 \equiv 0 \pmod{p}$.

219. Proposed by R. D. CARMICHAEL, Indiana University.

Determine whether it is possible for a polygon to have the number of its diagonals equal to a perfect fourth power.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

A solution of 399 by WM. CULLUM and a solution of 400 by LOUIS O'SHAUGHNESSY were received too late for credit in the May issue.

401. Proposed by R. D. CARMICHAEL, Indiana University.

Prove the validity of Borda's series:

$$\log(x+2) = 2\log(x+1) - 2\log(x-1) + \log(x-2) + 2\left[\frac{2}{x^3-3x} + \frac{1}{3}\left(\frac{2}{x^3-3x}\right)^3 + \frac{1}{5}\left(\frac{2}{x^3-3x}\right)^5 + \cdots\right].$$

SOLUTION BY A. M. HARDING, University of Arkansas.

We have

$$\log(x+2) - 2\log(x+1) + 2\log(x-1) - \log(x-2)$$

$$= \log\frac{(x-1)^2(x+2)}{(x+1)^2(x-2)} = \log\frac{x^3-3x+2}{x^3-3x-2} = \log\frac{1+\dfrac{2}{x^3-3x}}{1-\dfrac{2}{x^3-3x}}$$

$$= 2\left[\frac{2}{x^3-3x} + \frac{1}{3}\left(\frac{2}{x^3-3x}\right)^3 + \frac{1}{5}\left(\frac{2}{x^3-3x}\right)^5 + \cdots\right],$$

which proves the result.

It should be observed that the series converges only for values of x such that $x^3 - 3x > 2$. But $x^3 - 3x + 2 = (x + 1)^2(x - 2)$. Hence x must be greater than 2.

Also solved by A. G. CARIS, F. M. MORGAN, HORACE OLSON, ELIJAH SWIFT, S. W. REAVES, and T. M. BLAKESLEE.

402. Proposed by R. D. CARMICHAEL, Indiana University.

Obtain other series similar to that of Borda given in the preceding problem (No. 401).

SOLUTION BY S. W. REAVES, University of Oklahoma.

If in the well-known series

$$\text{(1)}\quad \log (n + 1) = \log n + 2\left[\frac{1}{2n + 1} + \frac{1}{3}\left(\frac{1}{2n + 1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n + 1}\right)^5 + \cdots\right]$$

we set

$$n = \frac{(x - 2k)(x + k)^2}{4k^3},$$

and therefore

$$n + 1 = \frac{(x + 2k)(x - k)^2}{4k^3}, \qquad \text{and} \qquad \frac{1}{2n + 1} = \frac{2k^3}{x^3 - 3k^2x},$$

we shall have a generalized Borda's series

$$\text{(2)}\quad \begin{aligned} \log (x + 2k) &= 2 \log (x + k) - 2 \log (x - k) + \log (x - 2k) \\ &\quad + 2\left[\frac{2k^3}{x^3 - 3k^2x} + \frac{1}{3}\left(\frac{2k^3}{x^3 - 3k^2x}\right)^3 + \frac{1}{5}\left(\frac{2k^3}{x^3 - 3k^2x}\right)^5 + \cdots\right]. \end{aligned}$$

For $k = 1$, (2) becomes Borda's series.

Again, if in (1) we set

$$n = \frac{(x - 3k)(x + k)^3}{16k^3x},$$

we shall have

$$\text{(3)}\quad \begin{aligned} \log (x + 3k) &= 3 \log (x + k) - 3 \log (x - k) + \log (x - 3k) \\ &\quad + 2\left[\frac{8k^3x}{x^4 - 6k^2x^2 - 3k^4} + \frac{1}{3}\left(\frac{8k^3x}{x^4 - 6k^2x^2 - 3k^4}\right)^3 + \cdots\right]. \end{aligned}$$

Also solved by F. M. MORGAN, T. M. BLAKESLEE, and A. M. HARDING.

GEOMETRY.

410. Proposed by A. H. HOLMES, Brunswick, Maine.

Given a focus and two tangents to an ellipse, prove that the locus of the foot of the normal corresponding to either tangent is a straight line.

SOLUTION BY A. M. HARDING, University of Arkansas.

Let SP and SP' be the fixed tangents, F the fixed focus, F' the other focus, and N the foot of the normal corresponding to SP. Draw FE and $F'E'$ perpendicular to SP, and FG and $F'G'$ perpendicular to SP'. Draw FF' cutting SP in L.

In any ellipse the product of the perpendiculars from the foci on any tangent is constant, *i. e.*, $EF \times E'F' = GF \times G'F' =$ constant.

Hence $E'F'/G'F' = GF/EF =$ constant, and the locus of F' is a fixed line SF'.

Now the lines PS, PF, PN, PF' form an harmonic pencil, and L, F, N, F' are harmonic points.

Hence the lines SL, SF, SN, SF' form an harmonic pencil.

But SL, SF, SF' are fixed. Therefore, SN is a fixed line.

Likewise the foot of the normal corresponding to the other tangent SP' lies on a fixed line which is the harmonic conjugate of SP' with respect to SF and SF'.

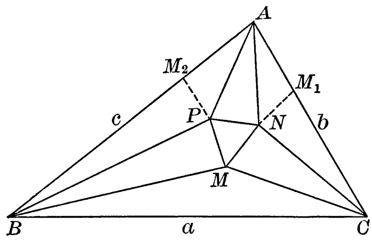
431. Proposed by E. M. MORGAN, Dartmouth College.

Trisect the angles of the triangle ABC and let the trisectors nearest each side meet in the respective points M, N, P. Prove by trigonometry that the triangle MNP is equilateral.

SOLUTION BY A. M. HARDING, University of Arkansas.

Let a, b, c denote the sides BC, CA, AB respectively. Construct the points M_1 and M_2 on CA and AB respectively, such that $CM_1 = CM$ and $BM_2 = BM$.

Then $MN = M_1N$ and $MP = M_2P$.



If 3α, 3β, 3γ represent the angles A, B, C we have

$$\overline{M_1N}^2 = \overline{AN}^2 + \overline{AM_1}^2 - 2AN \times AM_1 \cos\alpha,$$

or

$$\overline{MN}^2 = \overline{AN}^2 + (b - CM)^2 - 2AN(b - CM)\cos\alpha.$$

Also

$$PN^2 = \overline{AN}^2 + \overline{AP}^2 - 2AN \times AP\cos\alpha.$$

Hence

$$\overline{MN}^2 - \overline{PN}^2 = (b - CM)^2 - \overline{AP}^2 - 2AN(b - CM)\cos\alpha + 2AN \times AP\cos\alpha,$$

$$= (b - CM - AP)(b - CM + AP - 2AN\cos\alpha).$$

Now

$$CM = \frac{\sin\beta}{\sin(\beta+\gamma)}a, \qquad AP = \frac{\sin\beta}{\sin(\alpha+\beta)}c, \qquad AN = \frac{\sin\gamma}{\sin(\alpha+\gamma)}b.$$

Also

$$b = \frac{\sin 3\beta}{\sin 3\alpha}a, \qquad c = \frac{\sin 3\gamma}{\sin 3\alpha}a.$$

Hence

$$b - 2AN\cos\alpha - CM + AP$$

$$= \left[1 - \frac{2\sin\gamma\cos\alpha}{\sin(\alpha+\gamma)}\right]b + \frac{\sin\beta}{\sin(\alpha+\beta)}c - \frac{\sin\beta}{\sin(\beta+\gamma)}a$$

$$= \frac{\sin(\alpha-\gamma)}{\sin(\alpha+\gamma)}b + \frac{\sin\beta}{\sin(\alpha+\beta)}c - \frac{\sin\beta}{\sin(\beta+\gamma)}a$$

$$= \left[\frac{\sin(\alpha-\gamma)\sin 3\beta}{\sin(\alpha+\gamma)\sin 3\alpha} + \frac{\sin\beta\sin 3\gamma}{\sin(\alpha+\beta)\sin 3\alpha} - \frac{\sin\beta}{\sin(\beta+\gamma)}\right]a$$

$$= \left[\frac{\sin(\alpha-\gamma)\sin 3\beta}{\sin(\alpha+\gamma)} + \frac{\sin\beta\sin 3\gamma}{\sin(\alpha+\beta)} - \frac{\sin\beta\sin 3\alpha}{\sin(\beta+\gamma)}\right]\frac{a}{\sin 3\alpha}$$

$$= \{\sin(\alpha-\gamma)[3 - 4\sin^2(\alpha+\gamma)] + \sin\beta[3 - 4\sin^2(\alpha+\beta)] - \sin\beta[3 - 4\sin^2(\beta+\gamma)]\}\, a/\sin 3\alpha$$

$$= \{\sin(\alpha-\gamma)[3 - 4\sin^2(\alpha+\gamma)] - 4\sin\beta[\sin^2(\alpha+\beta) - \sin^2(\beta+\gamma)]\}\, a/\sin 3\alpha$$

$$= \{\sin(\alpha-\gamma)[3 - 4\sin^2(\alpha+\gamma)] - 2\sin\beta[\cos 2(\beta+\gamma) - \cos 2(\alpha+\beta)]\}\, a/\sin 3\alpha$$

$$= \{\sin(\alpha-\gamma)[3 - 4\sin^2(\alpha+\gamma)] - 4\sin\beta[\sin(\alpha-\gamma)\sin(\alpha+2\beta+\gamma)]\}\, a/\sin 3\alpha$$

$$= \sin(\alpha-\gamma)[3 - 4\sin^2(\alpha+\gamma) - 4\sin\beta\sin(\alpha+2\beta+\gamma)]\, a/\sin 3\alpha$$

$$= \sin(\alpha-\gamma)[1 + 2\cos 2(\alpha+\gamma) + 2\cos(\alpha+3\beta+\gamma) - 2\cos(\alpha+\beta+\gamma)]\, a/\sin 3\alpha.$$

Now

$$\cos(\alpha+3\beta+\gamma) = -\cos 2(\alpha+\gamma)$$

and

$$\cos(\alpha+\beta+\gamma) = \cos 60^\circ = \tfrac{1}{2}.$$

Hence $\quad b - 2AN\cos\alpha - CM + AP = 0$, and $MN = PN$.

Likewise $\quad MP = PN.$

Also solved by T. M. BLAKESLEE.

CALCULUS.

335. Proposed by W. R. LEBOLD, Cambridge, Ohio.

Let $\rho = F(\theta, \phi)$ be the equation in polar coordinates of a closed surface. Show that the volume of the solid bounded by the surface is equal to the double integral

$$\frac{1}{3}\iint \rho\cos\gamma\, d\sigma$$

extended over the whole surface, where $d\sigma$ represents the element of area, and γ the angle which the radius vector makes with the external normal. [Goursat-Hedrick, *Analysis*, p. 325, ex. 9.]

SOLUTION BY S. W. REAVES, University of Oklahoma.

We may take for element of volume dV the cone (or pyramid) having the pole for vertex and the element of area $d\sigma$ for base. The altitude of this cone is clearly $\rho \cos \gamma$, and hence its volume is $dV = \frac{1}{3}\rho \cos \gamma d\sigma$. Hence the volume V, which is the sum of all the elements of volume, is, by the theory of definite integrals,

$$V = \tfrac{1}{3}\iint \rho \cos \gamma d\sigma,$$

where the integration is to be taken over the entire surface.

344. Proposed by B. F. FINKEL, Drury College.

Solve the differential equation,

$$\frac{d^2y}{dx^2}(a^2 - x^2 - y^2) = \left(x\frac{dy}{ds} - y\frac{dx}{ds}\right)\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}.$$

SOLUTION BY A. M. HARDING, University of Arkansas.

The equation may be written in the form

$$a^2 - x^2 - y^2 = \left(x\frac{dy}{ds} - y\frac{dx}{ds}\right)\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \left(x\frac{dy}{ds} - y\frac{dx}{ds}\right)\rho,$$

where ρ is the radius of curvature at any point of the curve whose equation is sought.

Change to polar coördinates. Let ϕ be the angle the tangent makes with the initial line, and ψ the angle between the radius vector and the tangent, *i. e.*, $\phi = \theta + \psi$. Then

$$\frac{dr}{d\theta} = r \cot \psi$$

and

$$\rho = \frac{\frac{ds}{d\theta}}{\frac{d\phi}{d\theta}} = \frac{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}{1 + \frac{d\psi}{d\theta}} = \frac{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}{1 + \frac{d\psi}{dr}\cdot\frac{dr}{d\theta}} = \frac{r \csc \psi}{1 + r \cot \psi \frac{d\psi}{dr}}.$$

Also

$$\frac{dy}{ds} = \sin \phi, \qquad \text{and} \qquad \frac{dx}{ds} = \cos \phi.$$

Substituting, we obtain

$$a^2 - r^2 = r(\cos\theta \sin\phi - \sin\theta\cos\phi) \cdot \frac{r \csc\psi}{1 + r\cot\psi \dfrac{d\psi}{dr}}$$

$$= r\sin\psi \frac{r\csc\psi}{1 + r\cot\psi \dfrac{d\psi}{dr}} = \frac{r^2}{1 + r\cot\psi\dfrac{d\psi}{dr}},$$

whence

$$\cot\psi d\psi = \frac{2r^2 - a^2}{r(a^2 - r^2)} dr$$

or

$$\log\sin\psi = \tfrac{1}{2}\log\frac{1}{r^2(a^2 - r^2)} + \tfrac{1}{2}\log c,$$

i. e.,

$$\sin^2\psi = \frac{c}{r^2(a^2 - r^2)}.$$

But

$$\sin^2\psi = \frac{r^2}{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}.$$

Therefore

$$\frac{c}{r^2(a^2 - r^2)} = \frac{r^2}{r^2 + \left(\dfrac{dr}{d\theta}\right)^2},$$

whence

$$\frac{\sqrt{c}dr}{\sqrt{-cr^2 + a^2r^4 - r^6}} = d\theta.$$

Let $r^2 = x$ and obtain

$$\frac{\tfrac{1}{2}\sqrt{c}dx}{x\sqrt{-c + a^2x - x^2}} = d\theta.$$

Integrate and obtain

$$\tfrac{1}{2}\sin^{-1}\frac{a^2r^2 - 2c}{r^2\sqrt{a^4 - 4c}} = \theta + \left(\frac{\pi}{4} - \beta\right),$$

after putting r^2 for x. Whence

$$\frac{a^2r^2 - 2c}{r^2\sqrt{a^4 - 4c}} = \cos(2\theta - 2\beta) = \cos 2\theta\cos 2\beta + \sin 2\theta \sin 2\beta.$$

Change to rectangular coördinates

$$[a^2 - \sqrt{a^4 - 4c}\cos 2\beta]x^2 - 2\sin 2\beta\sqrt{a^4 - 4c}\,xy + [a^2 + \sqrt{a^4 - 4c}\cos 2\beta]y^2 = 2c.$$

Now introduce a new constant $k^4 = a^4 - 4c$. The equation may then be

written

$$[a^2 - k^2 \cos 2\beta]x^2 - 2k^2 \sin 2\beta \cdot xy + [a^2 + k^2 \cos 2\beta]y^2 = \frac{a^4 - k^4}{2}.$$

This is the equation of a system of conics, where the distance from the center to the focus is k, and where β is the angle between the major axis and the x-axis.

Also solved by ELIJAH SWIFT, who obtained the result $\phi = \int \frac{dr}{r\sqrt{cr^2(r^2 - a) - 1}}$, and by C. N. SCHMALL and who obtained $xy = c$.

The above solution by Professor HARDING is correct.

CALCULUS.

345. Proposed by C. N. SCHMALL, New York City.

Of all quadrilaterals which can be formed from four given sides, that which is inscriptible in a circle has the greatest area. [GOURSAT-HEDRICK, page 133, example 5.]

Proposer's Remark. Also show that when only three sides are known the length of the fourth side of the maximum quadrilateral is the root of a cubic equation.

SOLUTION BY THE PROPOSER.

Let the area $= u = \Delta ABC + \Delta ADC$. That is,

$$u = \tfrac{1}{2}(ab \sin \phi + cd \sin \psi). \tag{1}$$

Also

$$\overline{AC}^2 = a^2 + b^2 - 2ab \cos \phi$$
$$= c^2 + d^2 - 2cd \cos \psi.$$

Hence

$$a^2 + b^2 - 2ab \cos \phi = c^2 + d^2 - 2cd \cos \psi. \tag{2}$$

Now for a maximum we have from (1)

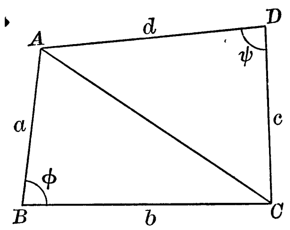
$$\frac{\partial u}{\partial \phi} = \frac{1}{2}\left(ab \cos \phi + cd \cos \psi \frac{\partial \psi}{\partial \phi}\right) = 0, \tag{3}$$

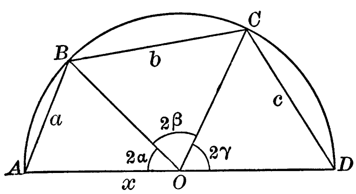
and from (2)

$$0 = 2ab \sin \phi - 2cd \sin \psi \frac{\partial \psi}{\partial \phi}. \tag{4}$$

Equating the values of $\partial\psi/\partial\phi$ obtained from (3) and (4), and reducing, we get

$$\frac{\cos \phi}{\sin \phi} = -\frac{\cos \psi}{\sin \psi}.$$





Hence $\tan\phi = -\tan\psi$, and $\phi + \psi = \pi$; that is, the opposite angles of the maximum quadrilateral are supplementary and hence the quadrilateral is cyclic.

Again, it is shown by elementary geometry, that when three sides of a maximum quadrilateral are given, the fourth side is the diameter of the circumscribing circle of the figure. Hence, to find this diameter AD, let $AD = x$, $AB = a$, $BC = b$, $CD = c$; and let the angles subtended by a, b, c, at the center be 2α, 2β, 2γ.

Then

$$\alpha + \beta + \gamma = \frac{\pi}{2} \tag{1}$$

and

$$\sin(\alpha + \beta + \gamma) = 1,$$

that is,

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma + 2\sin\alpha\sin\beta\sin\gamma = 1. \tag{2}$$

But $\sin\alpha = a/x$; $\sin\beta = b/x$; $\sin\gamma = c/x$. Substituting these values in (2), we have

$$x^3 - x(a^2 + b^2 + c^2) - 2abc = 0. \tag{3}$$

This cubic has two negative roots and one positive root, as may be seen by putting $x = 0, -a, -\infty$. The positive root is the value of x required. We can then describe a semicircle having this root as a diameter and place in it the chords a, b, c, in any order; for the equation (3) involves a, b, c, symmetrically.

Note.—The cubic (3) can also be derived as follows:

Draw the diagonals AC and BD. Then

$$AC = \sqrt{x^2 - c^2}, \qquad BD = \sqrt{x^2 - a^2}.$$

Hence, by *Ptolemy's Theorem* we have

$$\sqrt{x^2 - c^2} \cdot \sqrt{x^2 - a^2} = ac + bx,$$

whence

$$x^3 - (a^2 + b^2 + c^2)x - 2abc = 0,$$

as before.

Also solved by ELIJAH SWIFT, ELMER SCHUYLER, and A. M. HARDING.

MECHANICS.

273. Proposed by F. P. MATZ, Reading, Pa.

A person is placed on a perfectly smooth surface. How may he get off.

SOLUTION BY S. W. REAVES, University of Oklahoma.

He should throw some object, for example his hat, in the opposite direction to that in which he wishes to go. The reaction, or "recoil," will cause him to slide to the edge of the smooth surface.

284. Proposed by C. N. SCHMALL, New York, N. Y.

A cylindrical vessel standing upright on a horizontal plane is kept constantly full of water by an automatic device. Determine at what height in its side a small orifice should be made,

so that the water may spout through it to the greatest horizontal distance on the plane. What difference in the result when the cylinder stands on a shelf of known height above the plane?

SOLUTION BY J. B. SMITH, Hampden-Sidney, Va.

Let a be the height of the cylinder and b the height of the orifice above the plane. Let v_0 be the velocity of the jet at the orifice. Then

$$v_0 = \sqrt{2g(a-b)}, \quad x_0 = v_0 t, \quad \text{and} \quad y = b - \tfrac{1}{2}gt^2.$$

Hence,

$$y = b - \frac{x^2}{4(a-b)}$$

is the equation of the trajectory, the origin being at the base of the cylinder and the y-axis being the vertical through the orifice.

For $y = 0$, we have $x^2 = 4b(a - b)$. As x^2 is a maximum when x is, we have

$$\frac{d(x^2)}{db} = 4(a-b) - 4b = 0,$$

$$\therefore\ b = a/2.$$

Hence the range will be a maximum when the height of the orifice is half that of the cylinder.

If the cylinder stands on a shelf of height h above the plane, we wish to find the maximum range on the line $y = -h$. In this case we have

$$x^2 = 4(b+h)(a-b).$$

It follows that the range is a maximum when $b = (a - h)/2$. If $h = a$, the orifice must be at the bottom of the cylinder; if $h > a$, b would be negative, but as this is impossible, the greatest range for the given cylinder is obtained by making the orifice at the bottom.

Also solved by HORACE OLSON and B. LIBBY.

NUMBER THEORY.

200. Proposed by R. D. CARMICHAEL, Indiana University.

Find the general solution, in relatively prime integers, of the equation $x^2 + y^2 = z^4$.

SOLUTION BY ARTEMAS MARTIN, Washington, D. C.

Let $x = p^2 - q^2$, $y = 2pq$; then

$$x^2 + y^2 = (p^2 - q^2)^2 + (2pq)^2 = (p^2 + q^2)^2 = z^4.$$

Let $p = r^2 - s^2$, $q = 2rs$; then

$$p^2 + q^2 = (r^2 - s^2)^2 + (2rs)^2 = (r^2 + s^2)^2;$$

therefore

$$x^2 + y^2 = (p^2 + q^2)^2 = (r^2 + s^2)^4 = z^4.$$

Retracing,

$$x = (r^2 - s^2)^2 - (2rs)^2, \qquad y = 4rs(r^2 - s^2), \qquad z = r^2 + s^2;$$

therefore

$$[(r^2 - s^2)^2 - (2rs)^2]^2 + [4rs(r^2 - s^2)]^2 = (r^2 + s^2)^4,$$

where r and s may be any numbers prime to each other, one odd and the other even.

Let $r = 2$, $s = 1$, and we have

$$7^2 + 24^2 = 625 = 25^2 = 5^4.$$

Let $r = 3$, $s = 2$; then we get

$$119^2 + 120^2 = 169^2 = 13^4.$$

Let $r = 4$, $s = 1$, and we find

$$161^2 + 240^2 = 289^2 = 17^4.$$

203. Proposed by R. D. CARMICHAEL, Indiana University.

Find solutions in integers of the equation

$$2x^2 + 1 = 3y^2. \tag{1}$$

I. SOLUTION BY E. E. WHITFORD, New York City.

Let $2x = z$. Then

$$z^2 - 6y^2 = -2. \tag{2}$$

By inspection the solution in smallest positive integers is $z_1 = 2$, $y_1 = 1$. The problem will not lose in generality if the solution be limited to positive integers. To represent concisely all the positive solutions without exception and without repetition I have derived the following formula:

$$z + y\sqrt{6} = \frac{(z_1 + y_1\sqrt{6})^{2k+1}}{2^k},$$

where

$$z_1 = 2, \qquad y_1 = 1, \qquad k = 0, 1, 2, 3, \cdots.$$

Therefore

$$z = 2, \quad 22, \quad 218, \quad 2158, \quad \cdots,$$

$$y = 1, \quad 9, \quad 89, \quad 881, \quad \cdots.$$

Since the values of z are even each set gives a solution of equation (1).

$$x = 1, \quad 11, \quad 109, \quad 1079, \quad 10681, \quad \cdots,$$

$$y = 1, \quad 9, \quad 89, \quad 881, \quad 8721, \quad \cdots.$$

These results were obtained by Euler in his "Algebra," by de la Roche in his "Larismetique" (1520), who copied from the "Triparty" of Chuquet (1484); and probably by nearly everyone who tried, before decimal fractions came into common use, to find the approximate value of the square root of 6.

Equation (2) is a slightly generalized form of the Pell equation $x^2 - Ay^2 = 1$.

The Pell equation can be made an aid to the solution of all indeterminate equations of the second degree in two unknowns.*

II. SOLUTION BY RICHARD MORRIS, Rutgers College.

This equation is of the form $Dx^2 - Cy^2 = -H$ (see Chrystal's *Algebra*, Part II, page 450) and the integral values of x and y are dependent upon the expression of $\sqrt{CD}/D$ as a continued fraction, the value of x being equal to the numerator and that of y to the denominator of those convergents, where

$$(-1)^n M_n = -H.$$

Let n denote the number of the convergent, A_n the quotients, p_n/q_n the terms of the convergent, and M_n the $(n+1)$th rational divisor, and arrange these quantities in column form to exhibit the solutions. There will be an infinite number of solutions since $(-1)^n M_n = -1$ for all the odd convergents.

N	A_n	p_n	q_n	M_n
1	1	1	1	1
2	4	5	4	2
3	2	11	9	1
4	4	49	40	2
5	2	109	89	1
6	4	485	396	2
7	2	1,079	881	1
8	4	4,801	3,920	2
9	2	10,681	8,721	1

Thus $x = 1$, $y = 1$; $x = 11$, $y = 9$; $x = 109$, $y = 89$, etc., are integral solutions.

Also solved by C. E. GITHENS.

MISCELLANEOUS QUESTIONS.

EDITED BY R. D. CARMICHAEL.

Note.—This department is designed to furnish a forum for the discussion of live questions in the teaching of collegiate mathematics and of difficulties actually encountered by any one who may desire the reaction of other workers in the same field. It is believed that much benefit may accrue from the interchange of ideas along these lines, and it is hoped that a widespread interest will develop in maintaining the department.

EDITORS.

QUESTION.

12. In view of the notation used by Professor Slobin in his "Note on Certain Algebraic Equations" in the April issue, pages 113–115, a discussion would be desirable as to the best notation for complex roots in general, and in particular for eliminating the conspicuous ambiguities introduced by the notation above cited.

REPLIES.

7. What place should be given to the history of mathematics in courses for prospective high school teachers, and why?

* For an account of such solutions see "The Pell Equation," E. E. Whitford, *Publications of the College of the City of New York*, 1912, pages 60, 64, ff.

I. REMARKS BY G. A. MILLER, University of Illinois.

The history of mathematics is one of the most useful phases of applied mathematics since it enables us to measure an important type of intellectual development at various periods. A study of some of the applications of mathematics frequently makes particular subjects both easier and more interesting. It also establishes contact with other subjects and thus tends towards greater unity in the student's knowledge.

It seems to me that the prospective high school teacher should have a course in the history of mathematics extending over at least five hours per week for one semester. One difficulty is to find teachers in our universities and normal schools who are really prepared and willing to give such a course. After various trials I have come to the conclusion that a course in the history of elementary mathematics is very difficult and requires an unusual amount of preparation on the part of the teacher.

In the opening article of volume 14, February, 1914, of the *Bibliotheca Mathematica*, the noted mathematical historian, G. Eneström, considers the question of teaching the history of mathematics in the secondary schools. He emphasizes the fact that it is very difficult to secure accurate information as regards many of the most important historical questions in view of the many incorrect statements which appear even in the best histories of mathematics. Those who read only the English language are at a great disadvantage since no extensive history of mathematics has ever appeared in this language.

Notwithstanding these difficulties it seems to me that a course in the history of mathematics should be regarded as an essential element in the training of the high school teacher. Such a course should be one of the most useful means to impress the important fact that statements found in books must generally be subjected to a careful thought analysis in order to render them helpful for arriving at truths. It should also give a broader meaning to the various mathematical subjects and lead to a deeper interest by adding the dynamic element of development to the various mathematical concepts.

II. REMARKS BY W. T. STRATTON, Kansas State Agricultural College.

Every prospective high school teacher should be given an opportunity to become familiar with the history of the mathematics he will teach. The course need not be very extensive but should have at least two or three hours per week throughout the senior year devoted to it. This course should not only take up the history of the different subjects but should also go into the biographies of some of the best known mathematicians. Mathematics has a history that begins with the history of the race. It has held the attention of some of the greatest men of all times. The prospective mathematics teacher should be given a chance to receive the inspiration that will come to him by a study of the lives of these men. As he goes out to teach in the high schools he will find that new spirit and new life can often be put into his courses if he is able to point out

occasionally some interesting historical facts connected with the development of the subject in hand. Pupils, too, are always interested in men and if the subject is developed in connection with the facts concerning men who have made mathematical history they will likely look upon the subject with more interest and enthusiasm.

A course in the history of mathematics will enlarge the teacher's vision. He should be able to see the points of difficulty in the development of the subject by the race and should be better prepared to take the pupils over these difficulties. He will see the dependence and interdependence of arithmetic, algebra, and geometry. The three merge into each other so gradually that the dividing line cannot be determined. Nothing will help more than a good course in the history of the subject to give the teacher the proper attitude toward these divisions.

NOTES AND NEWS.

Under the Direction of Florian Cajori.

As previously announced in these columns, Professor Cajori will be in Europe during the coming year. During his absence, Professor W. D. Cairns, of Oberlin College, Ohio, will act as chairman of the committee on Notes and News, and all contributions of this character should be sent directly to him.

Dr. C. L. Bouton of Harvard University has been promoted to an associate professorship of mathematics.

The *Educational Review* for May contains an article on "Mathematics for culture" by Professor N. J. Lennes, of the University of Montana.

Dr. C. Küschke, formerly instructor at the University of California, where he took his doctorate, has accepted an instructorship in mathematics for the coming year at the University of Washington. He has recently been attending lectures at the University of Chicago.

Dr. G. N. Green, of the College of the City of New York, has accepted an instructorship in mathematics at Harvard University.

Mr. Lester Hill, formerly a graduate student at Columbia University and later at the University of Chicago, has been made assistant professor in mathematics at the University of Montana.

Professor N. J. Lennes, head of the department of mathematics at the University of Montana, will spend the coming summer abroad.

Bulletin No. 323, of the University of Texas, contains an article by Dr. Edward L. Dodd on "The error-risk of the median compared with that of the arithmetic mean."

Mr. Edward Kircher, graduate student at the University of Illinois, has been appointed to an instructorship in mathematics at the Massachusetts Institute of Technology.

Professor C. Runge, of the University of Göttingen, was elected president for 1914 of the German Mathematical Society. According to the most recent list this society has 769 members.

Henry Holt & Company announce that the following two mathematical books are in press: "Analytic geometry of space," by Virgil Snyder and C. H. Sisam; and "Plane analytical geometry," by L. W. Dowling and F. E. Turneaure.

John Wiley and Sons have just published "The theory of numbers," by Robert D. Carmichael as No. 13 of the series of mathematical monographs edited by Mansfield Merriman and Robert S. Woodward. Professor Carmichael is also author of No. 12 in this series entitled "The theory of relativity."

The Macmillan Company announces the publication of a new edition of Professor F. R. Moulton's *Introduction to Celestial Mechanics*. This has been revised and largely rewritten, but still preserves the general plan of the earlier edition.

Professor George N. Bauer has been appointed chairman of the department of mathematics of the college of science, literature, and the arts at the University of Minnesota for the year 1914–15.

An interesting article entitled "Elementary theorems on the hexahedron in non-euclidean geometry," by Dr. Arthur Ranum, of Cornell University, was published in the *Quarterly Journal of Pure and Applied Mathematics*, No. 178, 1914.

The annual dinner for instructors and fellows in mathematics, at the University of Chicago, given by the faculty of the department, occurred on the evening of May 15, 1914. The guest of honor was Dr. T. M. Putnam, of the University of California, who had just been promoted to the rank of associate professor and was on his way to Europe where he will spend the next half year on sabbatical vacation.

Professor F. Cajori's History of Elementary Mathematics is being translated into the Japanese language by Mr. Seiich Sunagawa, of the Ehimeken Female Normal College.

The Macmillan Company has just published a book which will be of interest to the readers of the Monthly. It is entitled "Dialogues concerning two new sciences," by Galileo Galilei, translated from the Italian and Latin into English by Henry Crew and Alfonse De Salvio. This will be reviewed in a later issue of the Monthly.

A new part of the German edition of the large mathematical encyclopedia was recently published. It bears the title "Ansätze und allgemeine Methoden der Systemmechanik." The French edition recently published a part entitled: "Fonctions sphériques; generalizations diverses des fonctions sphériques."

An error was made in the March issue of the MONTHLY, page 86, in the review of A. W. STAMPER's textbook on the Teaching of Arithmetic, where it was stated that the Austrian method of division is not given. A brief reference with explanation and incompleted division is found on pages 95–96 of this text.

The American Association for the Advancement of Science has appointed a Committee of One Hundred on scientific research, under the government, in the universities and in other institutions. Pure mathematics is represented on this committee by Professor G. A. MILLER, University of Illinois, and Professor E. H. MOORE, University of Chicago.

Special efforts are being made to secure as large a collection as possible of letters by and to LEONHARD EULER for publication in his complete works. In view of the fact that this publication is much more expensive than was expected, the committee in charge is anxious to secure more subscribers for these works, as well as more members of the Euler Society. The members of this society promise to contribute at least ten francs per year towards the expense of publishing Euler's complete works.

An "Elementary Theory of Equations" by Professor L. E. DICKSON, of the University of Chicago, has just been published by John Wiley and Sons. It contains a chapter of 13 pages on complex numbers, one of 33 pages on determinants, and one of 17 pages on resultants and discriminants. The whole book consists of 183 pages. A review of it will appear in a future number of the MONTHLY.

The next Summer meeting of the American Mathematical Society will be held at Providence, R. I., September 8, 9, 1914, in connection with the celebration of the one hundred and fiftieth anniversary of Brown University.

The fourth instalment of the National Geometry Syllabus, which is being reprinted in the *Irish Journal of Education,* appeared in the May issue. It begins with Section B, Logical Considerations.

The summer quarter of the University of Chicago opened June 15th with the usual large attendance. The registration in mathematics is especially strong in the advanced courses.

Dr. C. E. GITHENS, who is principal of the Union School at Wheeling, West Va., was recently elected superintendent of the public schools of that city. He has long been a contributor to the problem department of the MONTHLY.

Mr. J. D. ESHLEMAN has been elected to an instructorship in mathematics at Western Reserve University, and Mr. L. E. WILLIAMS to a similar position in the Georgia School of Technology. Both have been graduate students of the University of Chicago during the past year.

Professor L. E. DICKSON, of the University of Chicago, will spend the first semester, Autumn 1914, in residence at the University of California lecturing along the lines of his special work in number theory. He was at one time a member of the faculty of the University of California.

Miss MARY E. WELLS, formerly instructor at Mount Holyoke College and now a graduate student at the University of Chicago, has been appointed instructor in mathematics at Oberlin College for the coming year, during which time Dr. MARY E. SINCLAIR will be on leave of absence. A recent announcement that Dr. T. H. GRONWALL, of Princeton University, had been appointed to an assistant professorship in mathematics at Oberlin College was erroneous. His appointment was at Princeton University.

Teachers of mathematics who are interested in the wide scope of the applications of mathematics will find it interesting to read a paper by Professor I. J. SCHWATT, of the University of Pennsylvania, entitled "The Applications of the Calculus to the Medical Sciences," published in the *American Journal of the Medical Sciences*, March, 1914, page 409. Since this journal does not usually contain material of a mathematical nature, a special mention here seems desirable.

The firm of Allyn and Bacon has published a trigonometry by Professors E. J. WILCZYNSKI and H. E. SLAUGHT, of the University of Chicago. Among the interesting features of this text are the numerous well-selected applications, and the arrangement of the text in two distinct parts, the first containing only such theoretical development as is essential to the solution of triangles. A review of this book will appear in a later issue of the MONTHLY.

During the recent meeting of the Michigan Schoolmasters' Club at Ann Arbor an exhibit of early mathematical text-books from the university collections and the private libraries of Professors BEMAN, ZIWET and KARPINSKI was displayed in the library. A large number of photographs of mathematical manuscripts in European libraries and from the collection of Mr. G. A. Plimpton were loaned for the exhibit by Professor L. C. KARPINSKI.

The Council of Mathematical Teachers in New England has recently appointed a special committee on the status and welfare of mathematics in secondary schools, to investigate and report on current criticisms of high school mathematics. The membership of the committee is as follows: Mr. G. W. EVANS, Charlestown High School; Professor F. C. FERRY, Williams College; Mr. A. V. GALBRAITH, Middlesex School; Mr. F. P. MORSE, Revere High School; Mr. C. D. MESERVE, Newton High School; Professor S. E. SMITH, Mount Holyoke College; Miss H. R. PIERCE, Worcester High School; and Professor H. W. TYLER, Massachusetts Institute of Technology, *chairman*. Correspondence with persons having special information is invited.

The application of the new school law in Ohio will probably necessitate that those preparing to teach secondary school subjects and looking to certification by the state department of education, shall devote one fourth of their college work directly to preparation for teaching; fifteen semester hours to be of distinctly educational work including practice and observation courses, teachers training courses in secondary school subjects, psychology, history and principles of education, educational methods, etc., fifteen hours to be chosen from the fields of psychology, sociology, logic, ethics, philosophy, and education. Several of the Ohio colleges, among these Ohio State University, Oberlin College and Miami University, are enabling teachers to meet these new requirements at once by offering suitable courses in the summer sessions of 1914.

The annual meeting of the Society for the Promotion of Engineering Education was held at Princeton University, June 23–26, 1914. The papers were printed in advance in the Bulletin of the Society, so that only short abstracts were read at the meetings, thus leaving ample time for discussion. The prospectus stated that "It is necessary to have papers and discussions at a convention, but these are not more than half of the meeting. It would be worth while to come to Princeton, even if there were no papers, just to meet the one hundred and ninety-nine other teachers." At all events, it seems to be a decided step forward to print papers in advance and to spend less time in *reading* and more in *discussion* and in general intercourse.

At the mathematical conference held on April 3d at Ann Arbor, Professor THEODORE R. RUNNING, of the University of Michigan, presented a paper on "Graphical methods applied to the solution of algebraic equations," Miss KATHERINE G. HINE, of the Detroit Central High School, offered "Some class room suggestions," Mr. G. C. BARTOO, of the Jackson High School, discussed "The problem of reducing the number of failures in algebra," and Professor W. B. FORD, of the University of Michigan, talked on "The future of geometry." Next year the conference is planning to have a luncheon and informal discussion to replace part of the formal program. Professor L. C. KARPINSKI was elected chairman of the conference for 1914–1915 and Mr. EDWARD T. GEE, of the Detroit High School, secretary.

The editors of the MONTHLY acknowledge the receipt of the following journals on our exchange list: *School Science and Mathematics, The Mathematics Teacher, The Annals of Mathematics, The American Journal of Mathematics, Transactions of the American Mathematical Society, Bulletin of the Society for the Promotion of Engineering Education, Proceedings of the American Philosophical Society, The Monist, Popular Astronomy, Journal of the Royal Astronomical Society of Canada, The Mathematical Gazette, L'Intermédiaire des Mathématiciens, L'Enseignement Mathématique, Revue Semestrielle des Publications Mathématiques, Periodico di Matematica per L'Insegnamento Secondario, Bolletino di Biblografia e Storia delle Scienze Matematiche, Rendiconti del Circolo Matematico di Palermo, Giornale de*

Matematiche di Battaglini, Nieuw Archief voor Wiskunde, Wiskundige Opgaven met de Oplossingen, Revista de la Sociedad Matemática Española, Iberica- el Progreso de las Ciencias y de sus Aplicaciones, The Tôhoku Mathematical Journal, The Journal of the Indian Mathematical Society. It is the intention hereafter to mention more frequently in these notes such articles in our exchanges as are deemed to be of special interest to readers of the MONTHLY.

A mathematical colloquium will be held at Edinburgh, July 28–31, immediately following the Napier tercentenary celebration. There will be two lectures by Professor M. D'Ocagne, of Paris, on "Nomography"; four lectures by Professor H. W. Richmond, of Cambridge University, on "Infinity in geometry"; four lectures by Professor E. Cunningham, of Cambridge University, on "Critical studies of modern electrical theories"; and two lectures by Professor E. T. Whittaker, of the University of Edinburgh, on "The solution of algebraic and transcendental equations in the mathematical laboratory." The methods described by Professor Whittaker will be chiefly arithmetical, and the lectures will therefore be supplementary to those of Professor D'Ocagne on nomographic methods, which for most purposes are now said to be generally recognized as superior to the older graphical methods of calculation. The announcement states that "the establishment of nomography as a regular constituent of British mathematical teaching is much to be desired."

Readers of the MONTHLY are asked to give attention to the announcements of the various publishers of mathematical books found in this and other issues. These publishers are not only serving our readers by giving up-to-date information concerning new books on the historical and philosophical aspects of mathematics, as well as text-books in the various fields, but they are also contributing in a substantial manner toward the support of this journal.

Complete sets of the MONTHLY for the year 1913, being the first volume under the reorganization, are in constant demand, but the supply of certain numbers in that volume has been exhausted. These numbers are January and June, 1913. Since the last issue several extra copies of the January number have been sent in and now a few complete sets of volume xx are again available, and several other sets are complete except for the June number. Any one who can contribute an extra copy of either the January or the June number will confer a great favor upon the editors.

The next issue of the MONTHLY will appear in September. There are no issues in July and August. During the summer, Professor Finkel, the treasurer, will be at Boulder, Colorado, in charge of mathematics in the summer session of the University of Colorado. Business communications may still be addressed to him at Springfield, Missouri, where they will receive prompt attention. The Managing Editor will be in residence at the University of Chicago and may be addressed as usual at 5548 Kenwood Ave., Chicago, Ill.

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THE APPARENT SIZE OF A CUBE.

By A. M. HARDING, University of Arkansas.

1. The apparent size of any surface is defined as the solid angle of a cone whose vertex is at the point of sight and whose directrix is the curve bounding the surface. The solid angle, being measured by the area cut by the cone from a unit sphere whose center is at the point of sight, may vary from zero to 4π, the apparent size of the whole " sky." The apparent size of a surface as seen from a certain point of space is evidently a function of the position of the point of sight. Its value is given by the integral

$$\omega = \int \frac{\cos(n, r)}{r^2} dS \tag{1}$$

extended over the whole surface, where r is the radius vector from the point of sight to any point of the surface, dS any element of surface, and (n, r) the angle between r and the positive direction of the normal to the surface.

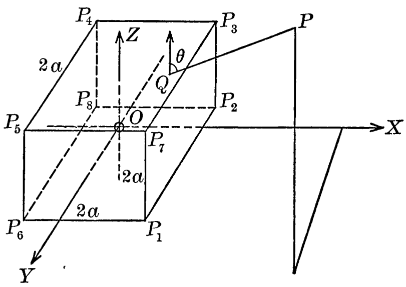


FIG. 1.

We shall take the center of the cube as origin and let the coördinate axes be parallel to the three edges of the cube. Let us consider first the solid angle subtended by the upper face of the cube, i. e., by the square $P_7P_3P_4P_5$. Let

$Q \equiv (\xi, \eta, a)$ be any point in this square and $P \equiv (x, y, z)$ any point in space. Then if $z > a$ the solid angle subtended at P by the square is given by formula (1) where $r = PQ$ and $\cos(n, r) \equiv \cos\theta = (z - a)/r$, and $dS = d\xi d\eta$.

Hence

$$\omega_1 = \int\int \frac{z-a}{r^3} d\xi d\eta = (z-a) \int_{-a}^{+a} \int_{-a}^{+a} \frac{d\xi d\eta}{[(x-\xi)^2 + (y-\eta)^2 + (z-a)^2]^{\frac{3}{2}}}.$$

After performing the integration we have

$$\omega_1 = \arctan\frac{(x-a)(y-a)}{(z-a)\Delta_7} + \arctan\frac{(x+a)(y+a)}{(z-a)\Delta_4} - \arctan\frac{(x+a)(y-a)}{(z-a)\Delta_5} - \arctan\frac{(x-a)(y+a)}{(z-a)\Delta_3}.$$

Likewise the solid angle subtended by the square $P_1P_2P_3P_7$ is given by

$$\omega_2 = \arctan\frac{(y-a)(z-a)}{(x-a)\Delta_7} + \arctan\frac{(y+a)(z+a)}{(x-a)\Delta_2} - \arctan\frac{(y+a)(z-a)}{(x-a)\Delta_3} - \arctan\frac{(y-a)(z+a)}{(x-a)\Delta_1};$$

and for the solid angle subtended by the square $P_5P_6P_1P_7$ we obtain

$$\omega_3 = \arctan\frac{(z-a)(x-a)}{(y-a)\Delta_7} + \arctan\frac{(z+a)(x+a)}{(y-a)\Delta_6} - \arctan\frac{(z+a)(x-a)}{(y-a)\Delta_1} - \arctan\frac{(z-a)(x+a)}{(y-a)\Delta_5},$$

where

$$\begin{aligned}
\Delta_1^2 &= (x-a)^2 + (y-a)^2 + (z+a)^2, & \Delta_4^2 &= (x+a)^2 + (y+a)^2 + (z-a)^2, \\
\Delta_2^2 &= (x-a)^2 + (y+a)^2 + (z+a)^2, & \Delta_5^2 &= (x+a)^2 + (y-a)^2 + (z-a)^2, \\
\Delta_3^2 &= (x-a)^2 + (y+a)^2 + (z-a)^2, & \Delta_6^2 &= (x+a)^2 + (y-a)^2 + (z+a)^2,
\end{aligned} \tag{2}$$
$$\Delta_7^2 = (x-a)^2 + (y-a)^2 + (z-a)^2.$$

2. The solid angle subtended by the entire cube will be the sum of one, two, or three of these solid angles according as one, two, or three faces of the cube are visible. Hence we must consider three different cases.

Case A. $\mathbf{x > a, \; -a < y < a, \; -a < z < a}$. In this case only one face is visible and $\omega = \omega_2$. Now by means of the formulas

$$\arctan(-m) = -\arctan(m) \quad \text{and} \quad \arctan(m) = \pi/2 - \arctan(1/m)$$

we can write the value of ω in the form

$$\omega = 2\pi - \left[\arctan \frac{(x-a)\Delta_1}{(a-y)(a+z)} + \arctan \frac{(x-a)\Delta_2}{(a+y)(a+z)} + \arctan \frac{(x-a)\Delta_3}{(a+y)(a-z)} + \arctan \frac{(x-a)\Delta_7}{(a-y)(a-z)} \right]. \tag{3}$$

Case B. **x > a, y > a, − a < z < a.** In this case two faces are visible and $\omega = \omega_2 + \omega_3$. If we combine the terms that contain the same radical by means of the formula

$$\arctan (m) + \arctan (n) = \arctan \frac{m+n}{1-mn}$$

we obtain, after further reduction,

$$\omega = 2\pi - \left[\arctan \frac{(a+z)\Delta_1}{(x-a)(y-a)} + \arctan \frac{(x-a)\Delta_2}{(y+a)(a+z)} + \arctan \frac{(x-a)\Delta_3}{(y+a)(a-z)} + \arctan \frac{(a-z)\Delta_7}{(x-a)(y-a)} + \arctan \frac{(y-a)\Delta_5}{(a-z)(x+a)} + \arctan \frac{(y-a)\Delta_6}{(x+a)(a+z)} \right]. \tag{4}$$

Case C. **x > a, y > a, z > a.** In this case three faces are visible and $\omega = \omega_1 + \omega_2 + \omega_3$. After combining terms as before we obtain

$$\omega = 2\pi - \left[\arctan \frac{(z+a)\Delta_1}{(x-a)(y-a)} + \arctan \frac{(x+a)\Delta_5}{(y-a)(z-a)} + \arctan \frac{(y+a)\Delta_3}{(z-a)(x-a)} + \arctan \frac{(z-a)\Delta_4}{(x+a)(y+a)} + \arctan \frac{(x-a)\Delta_2}{(y+a)(z+a)} + \arctan \frac{(y-a)\Delta_6}{(z+a)(x+a)} \right]. \tag{5}$$

It is not necessary to consider any more than these three cases since, for any given value of ω, the surfaces given by equations (3), (4), and (5) will be symmetrical with respect to the coördinate planes and the origin. Hence it is sufficient to study the formulas for one octant with different regions according to the number of faces visible. If the upper face alone is visible we can obtain the value of the solid angle by simply interchanging x and z in equation (3); if faces $P_1P_2P_3P_7$ and $P_2P_3P_4P_8$ are visible, simply change y to $-y$ in (4), etc.

It may be interesting to note that from any point of sight the cube will project into either a quadrilateral or a hexagon and in each of the equations (3), (4), (5) the radicals, which we have denoted by Δ, represent the distances from the point of sight to the vertices of that quadrilateral or hexagon.

3. The solid angle subtended by the cube may also be obtained from the relation $\omega = 2\pi - \sigma$ where σ is the length of that curve on the unit sphere which is polar to the intersection of the cone and the unit sphere. In this problem σ is the sum of the dihedral angles of the cone. Of course this cone is really a pyramid since its directrix is made up of straight lines.

If in each of the three cases we write the equations of the planes forming the lateral faces of the pyramids and then find the value of σ for each pyramid, the formula $\omega = 2\pi - \sigma$ leads to equations (3), (4), (5).

4. It may be interesting to note the behavior of these formulas as the point P moves away from the cube in any direction. If we change to polar coördinates and then express ω_1 as a power series in $1/r$, it will be found that the series begins with the term in $1/r^2$. In fact

$$\omega_1 = 4a^2 \cos (n, r) \times 1/r^2 + \cdots \text{ higher powers of } 1/r,$$

where (n, r) is the angle that the radius vector makes with the normal to the face. It will be found that ω_2 and ω_3 may be written in the same form. Hence we see that in each of the three cases (when one, two or three faces are visible) the apparent size of the cube will approach zero as the point of sight approaches infinity in any direction; and also that ωr^2 will approach

$$\Sigma(\text{area of face}) \times (\text{cosine of the angle between the radius vector and the normal to the face}).$$

In other words, as the point of sight approaches infinity in any direction, each of the faces of the cube acts more and more as if it were a single differential element $\cos \theta \, dS/r^2$.

5. If we combine certain terms in each of the equations (3), (4), (5) we are led to the following results: (*a*) when only one face is visible the solid angle may have any value between zero and 2π, being equal to 2π only when the point of sight lies on the face of the cube; (*b*) when only two faces are visible the solid angle lies between zero and π, being equal to π only when the point of sight lies on one edge of the cube; (*c*) when three faces are visible the solid angle lies between zero and $\pi/2$, being equal to $\pi/2$ only when the point of sight is at one corner of the cube.

If we let ω be constant in equations (3), (4), (5) we have in each case the equation of a surface. The apparent size of the cube will not vary as long as the point of sight remains on this surface. We propose now to study the forms of these surfaces for different values of ω.

6. Let us assume all space outside the cube to be divided into the following regions: (*a*) six prisms bounded by the six faces of the cube produced and extending, one from each face of the cube, to infinity; (*b*) twelve wedge-shaped regions extending, one from each edge, to infinity and bounded by the two faces which are perpendicular to this edge and the two faces which pass through the edge; (*c*) eight three-edged regions extending, one from each corner of the cube, to infinity and bounded by the three faces produced which meet in the given corner.

Then if ω has a constant value, such that all three of our surfaces exist, we have:

Case A. six distinct surfaces, one in each of the regions (a), symmetrical in pairs with respect to the three lines joining the center of the cube to the mid-points of the faces and symmetrical in pairs with respect to the origin;

Case B. twelve distinct surfaces, one in each of the regions (b), symmetrical in pairs with respect to the lines joining the mid-points of opposite edges of the cube and symmetrical in pairs with respect to the origin;

Case C. eight distinct surfaces, one in each of the regions (c), symmetrical in pairs with respect to the diagonals of the cube and symmetrical in pairs with respect to the origin.

It is not necessary to study each of these twenty-six different surfaces since any one of them may be obtained from one of the equations (3), (4), (5) by various permutations of letters and signs. For simplicity we shall refer to the surface from which only one face is visible as surface I or $F_1(x, y, z) = 0$; the surface from which two faces are visible we shall call surface II or $F_2(x, y, z) = 0$; and the surface from which three faces are visible we shall call surface III or $F_3(x, y, z) = 0$. Their equations are given by (3), (4), (5).

7. We propose now to obtain the equations of the curves in which the planes of the faces produced intersect surfaces I, II, and III. Of course it must be understood in all that follows that $\omega < \pi/2$, otherwise some of the surfaces may not exist. The plane $y = a$ separates the region of space in which surface I lies from that in which surface II is found. It can be shown easily that both these surfaces are cut by the plane $y = a$ in the same curve, whose equation is

$$\omega = \pi - \left[\arctan\frac{(x-a)\Delta_2}{2a(a+z)} + \arctan\frac{(x-a)\Delta_3}{2a(a-z)} \right]. \tag{6}$$

Likewise surfaces II and III are both cut by the plane $z = a$ in the same curve, whose equation is

$$\omega = \pi - \left[\arctan\frac{2a\Delta_1}{(x-a)(y-a)} + \arctan\frac{(x-a)\Delta_2}{2a(y+a)} + \arctan\frac{(y-a)\Delta_6}{2a(x+a)} \right]. \tag{7}$$

Hence we see that although we have twenty-six different surfaces defined by twenty-six distinct equations, they come together in pairs on the faces of the cube produced and form a single surface completely surrounding the cube and continuous everywhere.

8. Let us see whether any pair of these surfaces have a common tangent plane at any point of their curve of intersection. The direction cosines of the normal to the tangent plane to any surface are proportional to $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial z$, where $F(x, y, z) = 0$ is the equation of the surface. These partial derivatives may be easily obtained from equations (3), (4), (5), and it is not necessary to write their values here.

Since surfaces I and II meet on the plane $y = a$ we must investigate the values

of $\partial F_1/\partial x$, $\partial F_1/\partial y$, $\partial F_1/\partial z$, $\partial F_2/\partial x$, $\partial F_2/\partial y$, $\partial F_2/\partial z$ for $y = a$. If at any point on the curve of intersection we find the first three of these derivatives respectively equal to or proportional to the last three then the two surfaces have a common tangent plane at that point. It can be easily shown that $\partial F_1/\partial x \equiv \partial F_2/\partial x$ and $\partial F_1/\partial z \equiv \partial F_2/\partial z$, and $\partial F_1/\partial y \not\equiv \partial F_2/\partial y$. Hence the two surfaces do not have a common tangent plane at *every* point of their curve of intersection. Let us see whether they have a common tangent plane at *any* point of their curve of intersection. If $\partial F_1/\partial y = \partial F_2/\partial y$ it easily follows that

$$(x + a)(a - z)\Delta_1 - (x - a)(a - z)\Delta_6 = (x - a)(a + z)\Delta_5 - (x + a)(a + z)\Delta_7.$$

After a rather lengthy reduction it can be shown that this equation is not possible unless $x = a$ and $z = \pm a$, i. e., unless the curve of intersection passes through one corner of the cube. This cannot happen except for $\omega = \pi/2$, which will be considered later. Hence, if $\omega < \pi/2$, the surfaces I and II have no common tangent plane at any point of their curve of intersection.

Likewise it can be shown that, if $\omega < \pi/2$, surfaces II and III have no common tangent plane at any point of their curve of intersection.

9. We have said that each of the surfaces I, II, and III is symmetrical with respect to a certain line which we shall call the axis of the surface. The X-axis is the axis of surface I; the line joining the origin to the point $(a, a, 0)$ is the axis of surface II; and surface III has for its axis the line joining the origin to the point (a, a, a), the corner of the cube. We propose to show that each axis is a normal to its surface.

The axis of surface I will cut it at some point on the X-axis. Hence we must investigate the values of $\partial F_1/dx$, $\partial F_1/\partial y$, $\partial F_1/\partial z$ when $y = z = 0$. We find

$$\Delta_1 = \Delta_2 = \Delta_3 = \Delta_7, \quad \text{and} \quad \frac{\partial F_1}{\partial x} = \frac{-8a^2}{\Delta_1[(x - a)^2 + a^2]}, \quad \frac{\partial F_1}{\partial y} = 0, \quad \frac{\partial F_1}{\partial z} = 0.$$

Hence the direction cosines of the normal at the point $(x, 0, 0)$ are $1, 0, 0$. In other words the X-axis is the normal.

The axis of surface II will cut the surface at some point on the line $y = x$, $z = 0$. If we put $y = x$ and $z = 0$ we find $\Delta_1 = \Delta_7$, $\Delta_2 = \Delta_3 = \Delta_5 = \Delta_6$, and $\partial F_2/\partial x = \partial F_2/\partial y \neq 0$, and $\partial F_2/\partial z = 0$. Hence the direction cosines of the normal at this point are proportional to $1, 1, 0$; i. e. $\lambda = 1/\sqrt{2}$, $\mu = 1/\sqrt{2}$, $\nu = 0$. Hence the axis is the normal.

The axis of surface III will cut the surface at some point where $z = y = x$. If we put $z = y = x$ we find $\Delta_1 = \Delta_3 = \Delta_5$ and $\Delta_2 = \Delta_4 = \Delta_6$. Also $\partial F_3/\partial x = \partial F_3/\partial y = \partial F_3/\partial z$. Hence the direction cosines of the normal are $1/\sqrt{3}$, $1/\sqrt{3}$, $1/\sqrt{3}$. In other words the line $y = z = x$ is the normal.

10. In order to make the surfaces more vivid geometrically we shall find the coördinates of several points on them. Let us first find the length of the intercept of each surface on its axis.

Surface I. Put $z = y = 0$ in equation (3) and obtain

$$\frac{\pi}{2} - \frac{\omega}{4} = \arctan \frac{\sqrt{(x-a)^2 + 2a^2}}{a^2}.$$

Solving for x we have $x = a[1 \pm \sqrt{\csc(\omega/4) - 1}]$. We must take the positive sign in order to be on the positive side of the face. Hence

$$x = a\left[1 + \sqrt{\csc \frac{\omega}{4} - 1}\right]. \tag{8}$$

This gives us the length of $OA_z = OA_y = OA_x$. (Fig. 2.)

Surface II. Putting $z = 0$ and $y = x$ in equation (4), we obtain

$$\omega = 2\pi - \left[2 \arctan \frac{a\Delta_1}{(x-a)^2} + 4 \arctan \frac{(x-a)\Delta_2}{a(x+a)}\right], \quad \Delta_1{}^2 = 2x^2 - 4ax + 3a^2,$$

or

$$\frac{\pi}{2} - \frac{\omega}{4} = \frac{1}{2} \arctan \frac{a\Delta_1}{(x-a)^2} + \arctan \frac{(x-a)\Delta_2}{a(x+a)}, \quad \Delta_2{}^2 = 2x^2 + 3a^2.$$

Let

$$\varphi = \arctan \frac{a\Delta_1}{(x-a)^2}; \quad \text{then} \quad \tan \varphi = \frac{a\Delta_1}{(x-a)^2}$$

and

$$\frac{\varphi}{2} = \arctan \frac{1 - \cos \varphi}{\sin \varphi} = \arctan \frac{a}{\Delta_1}.$$

Hence

$$\tan\left(\frac{\pi}{2} - \frac{\omega}{4}\right) = \arctan \frac{a}{\Delta_1} + \arctan \frac{(x-a)\Delta_2}{a(x+a)},$$

then

$$\tan\left(\frac{\pi}{2} - \frac{\omega}{4}\right) = \cot \frac{\omega}{4} = \frac{a^2(x+a) + (x-a)\Delta_1\Delta_2}{a(x+a)\Delta_1 - a(x-a)\Delta_2}.$$

Rationalizing the denominator, we obtain

$$\cot \alpha = \frac{(x^2 + a^2)\Delta_1 + (x^2 - a^2)\Delta_2}{2a^2x}, \quad \text{where} \quad \alpha = \frac{\omega}{4},$$

or

$$2a^2x \cot \alpha - (x^2 - a^2)\Delta_2 = (x^2 + a^2)\Delta_1.$$

If we rationalize this equation we obtain

$$\begin{aligned}(1-c)^2x^8 &- 4(1-c)^2ax^7 + 2(1-c)(3-5c)a^2x^6 - 4(1-c)(3-4c)a^3x^5 \\ &+ 3(1-c)(5-7c)a^4x^4 - 4(1-c)(3-5c)a^5x^3 + 4(3-6c+4c^2)a^6x^2 \\ &- 4(1-c)(1-2c)a^7x - 2(1-c)(1+2c)a^8 = 0,\end{aligned} \tag{9}$$

where c is written in place of $\cos 2\alpha = \cos(\omega/2)$ for simplicity. This equation

has at least one positive root greater than a and this root can be found after c is known. It may happen for certain values of c that the equation has more than one positive root greater than a, in which case we must choose the root which satisfies the equation

$$\cot \frac{\omega}{4} = \frac{(x^2 + a^2)\Delta_1 + (x^2 - a^2)\Delta_2}{2a^2x}.$$

If x_1 is the required root of equation (9) then the length of the intercept is $x_1\sqrt{2}$. This gives us the length of $OB_z = OB_y = OB_x$ in Fig. 2.

Surface III. Putting $z = y = x$ in equation (5), we obtain

$$\omega = 2\pi - \left[3 \arctan \frac{(x+a)\Delta_1}{(x-a)^2} + 3 \arctan \frac{(x-a)\Delta_2}{(x+a)^2}\right]$$

where

$$\Delta_1^2 = 2(x-a)^2 + (x+a)^2; \quad \Delta_2^2 = 2(x+a)^2 + (x-a)^2.$$

From this equation we obtain, after a rather lengthy reduction,

$$x = \pm a\sqrt{\tan\left(\frac{\pi}{6} - \frac{\omega}{12}\right) \cot \frac{\omega}{4}}. \tag{10}$$

Hence $OC = x\sqrt{3} = a\sqrt{3 \tan (\pi/6 - \omega/12) \cot (\omega/4)}$.

In the case of surfaces I and III we have found the length of the intercept as an explicit function of ω, but in the case of surface II we did not obtain a simple result. The reason for this is obvious geometrically. If we are considering only one face of the cube, surface I is symmetrical with respect to the plane of this face and hence we obtain $x - a = \pm f(\omega)$. Likewise when we are considering three faces the surface is symmetrical with respect to the origin and we obtain $x = \pm \varphi(\omega)$. This symmetry does not exist in the case of surface II. The solid angle will not be the same for two points on the axis such that $x = \pm \psi(\omega)$ if we are looking at the *same* two faces. That is, equation (9) should not have two roots of the form $x = \pm \psi(\omega)$. It can be shown easily that equations (8), (9), and (10) hold for limiting values of ω. When $\omega = 0$, each equation gives $x = \infty$; when $x = a$ we have $\omega = 2\pi$, π, and $\pi/2$ respectively.

11. We have seen that surfaces I and II are both cut by the plane $y = a$ in the curve

$$\pi - \omega = \arctan \frac{(x-a)\Delta_2}{2a(a+z)} + \arctan \frac{(x-a)\Delta_3}{2a(a-z)}. \tag{11}$$

The arc Q_1M_z (Fig. 2) represents the upper half of this curve. If we find dz/dx from (11) it can be easily shown that the curve Q_1M_z cuts the XY plane at right angles for every value of ω, and that it makes with the edge SP_7 (produced) an acute angle whose value depends upon ω. In fact this angle varies from zero to $\pi/2$ as ω varies from $\pi/2$ to zero.

Putting $z = 0$ in (11), we obtain $\pi - \omega = 2 \arctan (x-a)\Delta_2/2a^2$, whence

$$\cot (\omega/2) = \tan (\pi/2 - \omega/2) = (x-a)\Delta_2/2a^2.$$

Then

$$\cot^2(\omega/2) = (x-a)^2[(x-a)^2 + 5a^2]/4a^4$$

or

$$(x-a)^4 + 5a^2(x-a)^2 - 4a^4\cot^2(\omega/2) = 0,$$

whence

$$x - a = HM_z = a\sqrt{\frac{-5 + \sqrt{25 + 16\cot^2(\omega/2)}}{2}}. \tag{12}$$

Hence, when ω is known, we have the lengths of the lines $HM_z = HM_z' = LM_y = LM_y' = SM_x = SM_x'$.

Now putting $z = a$ in (11), we obtain after reduction

$$x - a = Q_1P_7 = 2a\sqrt{\csc\omega - 1}. \tag{13}$$

Hence we have the lengths of the lines $Q_1P_7 = Q_2P_7 = Q_3P_7$.

12. Let us now study the curve of intersection of surfaces II and III. This is the curve Q_1Q_2 (Fig. 2). Its equation is

$$\pi - \omega = \arctan\frac{2a\Delta_1}{(x-a)(y-a)} + \arctan\frac{(x-a)\Delta_2}{2a(y+a)} + \arctan\frac{(y-a)\Delta_6}{2a(x+a)}. \tag{14}$$

If we find the value of dy/dx from (14) we can easily show that the curve cuts the diagonal (produced) of the upper face at right angles, and that it makes with the positive direction of P_7Q_1 an obtuse angle whose size depends on the value of ω.

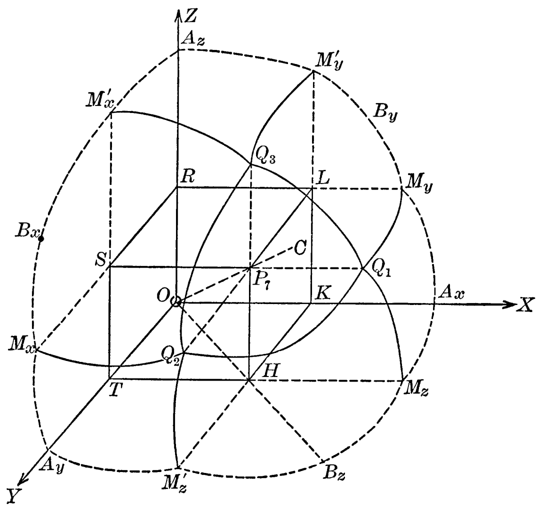


FIG. 2.

In fact as x approaches a, dy/dx approaches -1; and as x approaches ∞, dy/dx approaches $-\infty$. Hence this obtuse angle varies from $3\pi/2$ to $\pi/2$.

We can find the length of P_7Q_1 by putting $y = a$ in equation (14). This gves the same result as before.

13. Curvature properties. Let $U=\partial F/\partial x$, $V=\partial F/dy$, $W=\partial F/\partial z$, $u=\partial^2F/\partial x^2$, $v=\partial^2F/\partial y^2$, $w=\partial^2F/\partial z^2$, $u'=\partial^2F/\partial y\partial z$, $v'=\partial^2F/\partial z\partial x$, $w'=\partial^2F/\partial x\partial y$. Also $H=u+U(Uu'-Vv'-Ww')/VW$, $K=v+V(Vv'-Ww'-Uu')/WU$, $L=w+W(Ww'-Uu'-Vv')/UV$. Then the principal radii of curvature at any point are given by the quadratic equation

$$\frac{U^2}{H-Q/\rho}+\frac{V^2}{K-Q/\rho}+\frac{W^2}{L-Q/\rho}=0, \quad \text{where} \quad Q^2=U^2+V^2+W^2.$$

From the results already obtained it is geometrically reasonably certain that the parts of the surface between the curves of discontinuous slope are everywhere convex outward. This can be proved by showing that for any given point (x_1, y_1, z_1) the two values of ρ obtained from the equation above are both negative.

After finding the values of $\partial^2F/\partial x^2$, $\partial^2F/\partial y^2$, $\partial^2F/\partial z^2$ it can be easily shown that $\partial^2F/\partial x^2+\partial^2F/\partial y^2+\partial^2F/\partial z^2=0$. That is, the apparent size corresponds to a magnetic potential function.

14. We have assumed up to this point that ω is less than $\pi/2$. Let us now consider the forms of the different surfaces for other values of ω.

$\omega=\pi/2$. In this case surface III reduces to a mere point, the corner of the cube (See Fig. 3), and we have to study only surfaces I and II. The edge SP_7 is now tangent to both surfaces as we have seen before (Art. 8). Putting $\omega=\pi/2$ in (8) we find $OA_x=a(1+\sqrt{\csc(\pi/8)-1})$. The equation of the curve of intersection of surfaces I and II now becomes

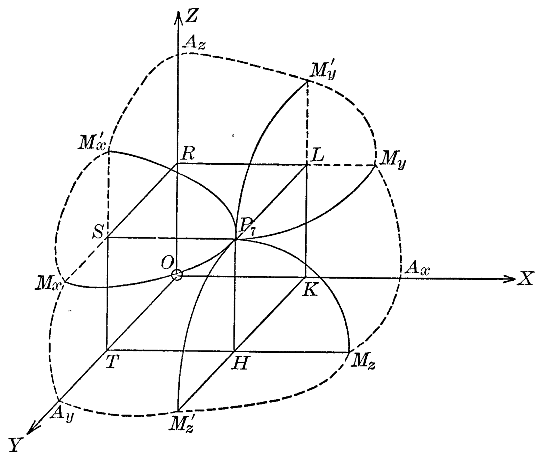


FIG. 3.

$$\frac{\pi}{2}=\arctan\frac{(x-a)\Delta_2}{2a(a+z)}+\arctan\frac{(x-a)\Delta_3}{2a(a-z)}.$$

From this equation we obtain after reduction, $4a^2(a^2-z^2)=(x-a)^2\Delta_2\Delta_3$. Squaring both sides, we obtain, after reduction,

$$(x-a)^6+2(x-a)^4(3a^2+z^2)+(x-a)^2(a^2-z^2)^2-4a^2(a^2-z^2)^2=0. \quad (15)$$

This is the algebraic equation of the curve of intersection. Putting $z = 0$ we have

$$(x-a)^6 + 6a^2(x-a)^4 + a^4(x-a)^2 - 4a^6 = 0, \quad \text{or} \quad (x-a)^4+5a^2(x-a)^2-4a^4=0,$$

after dividing by $(x - a)^2 + a^2$. This gives

$$HM_z = (x - a) = a\sqrt{\frac{-5+\sqrt{41}}{2}} = 0.8a,$$

approximately.

If we put $\omega = \pi/2$ in equation (12) we obtain the same result.

Putting $z = a$ in (15) we have $(x - a)^6 + 8a^2(x - a)^4 = 0$, whence $P_7Q_1 = x - a = 0$. This agrees with (13) for $\omega = \pi/2$. It can be easily shown from (15) that $dz/dx = 0$ for $z = x = a$ and $dz/dx = \infty$ for $z = 0$. Hence the point Q_1 is at P_7 and the curve P_7M_z is tangent to the edge SP_7 at P_7.

$\pi/2 < \omega < \pi$. In this case surfaces I and II still exist and OA_x is still given by (8) and HM_z by (12). The equation of the curve of intersection of the surfaces I and II, equation (11), is not satisfied by $x = a$ for any value of z, nor by $z = a$ for any value of x. In other words the curve of intersection approaches the point P_7 but never reaches it. The curve is still perpendicular to the XY plane at M_z.

$\omega = \pi$. The equation of surface II now reduces to $x = a$, $y = a$, the edge of the cube, and the curve of intersection of surfaces I and II reduces to $x = a$. This is the equation of the edge P_7P_1 with the end points left off, since z cannot be equal to a. The length of OA_x is now equal to $a[1 + \sqrt{\sqrt{2} - 1}]$.

$\pi < \omega < 2\pi$. In this case surface I alone exists and OA_x is still given by (8). This surface gradually closes down on the face of the cube as ω approaches $\pi/2$.

15. Forms of different level surfaces. By "level surfaces" we mean surfaces from which the cube has the same apparent size.

A. $0 < \omega < \pi/2$. In this case the surface consists of twenty-six distinct parts which form one continuous surface completely surrounding the cube and which approaches a sphere as ω approaches 0.

B. $\omega = \pi/2$. Eight of these surfaces now reduce to mere points, and the remaining eighteen form a continuous surface completely surrounding the cube and touching it at its corners.

C. $\pi/2 < \omega < \pi$. Only eighteen surfaces exist and they form a surface continuous everywhere except at the corners of the cube. This is of the same form as case *B* except the corners are left out.

D. $\omega = \pi$. Twelve of the surfaces reduce to lines (the edges of the cube), and the remaining six form a surface continuous everywhere except at the corners of the cube and touching the cube along its twelve edges.

E. $\pi < \omega < 2\pi$. Only six surfaces exist; one opposite each face. They form a surface completely surrounding the cube except that each edge is a line of discontinuity and each corner is a point of discontinuity.

RESIDUES OF CERTAIN SUMS OF POWERS OF INTEGERS.

By T. M. PUTNAM, University of California.

In Vol. XXXI, pages 329–333 of the *Quarterly Journal of Mathematics*, GLAISHER showed that, if p be an odd prime, and

$$H_n = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \cdots + \frac{1}{(p-1)^n},$$

then, when n is not a multiple of $p - 1$, $H_n \equiv 0 \bmod p$; and when n is a multiple of $p - 1$, $H_n \equiv -1 \bmod p$. By using Fermat's Theorem we have

$$1/a^n \equiv a^{p-n-1} \bmod p,$$

so that, putting $p - n - 1 = r$, we obtain

$$H_n \equiv 1 + 2^r + 3^r + \cdots + (p-1)^r \bmod p.$$

Hence the sum of the rth powers of the first $p - 1$ integers is congruent to -1, or zero, according as r is, or is not, a multiple of $p - 1$.

These results of Glaisher, as well as some others, may be obtained in an elementary way by starting from the identity,[1]

$$p^{r+1} - p = {}_{r+1}C_1 S_r + {}_{r+1}C_2 S_{r-1} + \cdots + {}_{r+1}C_r S_1,$$

where $S_r = 1 + 2^r + 3^r + \cdots + (p-1)^r$.

By taking, successively, $r = 1, 2, 3, \cdots p - 2$, we obtain

$$S_r \equiv 0 \bmod p.$$

If $r = p - 1$, $a^r \equiv a^{p-1} \equiv 1 \bmod p$; hence $S_{p-1} \equiv p - 1 \equiv -1 \bmod p$.

Furthermore since $S_{r'} \equiv S_r \bmod p$ whenever $r' \equiv r \bmod p - 1$, the residues of this series are known for every power r.

Since $a^{2k} \equiv (p-a)^{2k} \bmod p$,

$$S_{2k} \equiv 1^{2k} + 2^{2k} + 3^{2k} + \cdots + 3^{2k} + 2^{2k} + 1^{2k};$$

hence denoting by T_r the sum

$$1^r + 2^r + 3^r + \cdots + \left(\frac{p-1}{2}\right)^r$$

we obtain

$$2T_{2k} \equiv S_{2k} \bmod p.$$

Therefore, when $2k$ is not a multiple of $p - 1$, $T_{2k} \equiv 0 \bmod p$; and when $2k$ is a multiple of $p - 1$

$$T_{2k} \equiv -\tfrac{1}{2} \bmod p.$$

The determination of the residues of T_{2k-1} may also be made to depend upon

[1] C. Smith, *Treatise on Algebra*, 5th edition, p. 404.

the original identity. If the latter be written in terms of the T's it takes the form

$$\left(\frac{p+1}{2}\right)^{r+1} - \left(\frac{p+1}{2}\right) = {}_{r+1}C_1 T_r + {}_{r+1}C_2 T_{r-1} + \cdots + {}_{r+1}C_r T_1$$

or

$${}_{r+1}C_1 T_1 + {}_{r+1}C_2 T_2 + \cdots + {}_{r+1}C_r T_r \equiv \frac{1}{2^{r+1}} - \frac{1}{2} \bmod p.$$

By letting $r = 1, 2, 3, \cdots$, successively, the residues of T_1, T_2, T_3, etc., may be computed. It will be found as shown above that for any even value of r the residue of T_r is zero. The first seven odd values of r give the following results.

$$T_1 \equiv -\frac{1}{2^3}, \qquad T_3 \equiv \frac{1}{2^6}, \qquad T_5 \equiv -\frac{1}{2^7}, \qquad T_7 \equiv \frac{17}{2^{11}},$$

$$T_9 \equiv -\frac{31}{2^{11}}, \qquad T_{11} \equiv \frac{691}{2^{14}}, \qquad T_{13} \equiv -\frac{5461}{2^{15}}, \qquad \text{each taken mod } p.$$

These results hold for any prime p.

From a formula given by Glaisher in the *Quarterly Journal*, volume XXXII, page 279, a general expression for these residues may be derived. The formula is

$$\frac{1}{2^{2i+1}} + \frac{1}{4^{2i+1}} + \cdots + \frac{1}{(p-1)^{2i+1}} \equiv (-1)^{h-i}(2^{2h-2i} - 1)\frac{B_{h-i}}{2h-2i} \quad \bmod p,$$

where $2h = p - 1$ and B_{h-i} is the Bernoulli number of rank $h - i$.

Remembering that $1/a^{2i+1} \equiv a^{p-2i-2} \bmod p$, and taking out $1/2^{2i+1}$ from the left member, we get

$$\frac{1}{2^{2i+1}}\left\{1 + 2^{p-2i-2} + 3^{p-2i-2} + \cdots + \left(\frac{p-1}{2}\right)^{p-2i-2}\right\}$$
$$\equiv (-1)^{h-i}\cdot(2^{2h-2i} - 1)\frac{B_{h-i}}{2h-2i} \quad \bmod p.$$

Letting $h - i = k$, or $p - 2i - 2 = 2k - 1$, and $2i + 1 = p - 2k$, we get

$$T_{2k-1} \equiv (-1)^k \cdot 2^{p-2k}(2^{2k} - 1)\cdot\frac{B_k}{2k} \equiv (-1)^k\frac{2^{2k}-1}{2^{2k}}\cdot\frac{B_k}{k} \quad \bmod p.$$

This formula holds for $k = 1, 2, \cdots, \frac{1}{2}(p-1)$, but for $k = \frac{1}{2}(p-1)$ the denominator of B_k is divisible by p, so that it should be written

$$T_{p-2} \equiv (-1)^{\frac{1}{2}(p-1)}\cdot\frac{2^{p-1}-1}{2^{p-1}p}\cdot\frac{pB_{\frac{1}{2}(p-1)}}{\frac{1}{2}(p-1)}$$
$$\equiv (-1)^{\frac{1}{2}(p+1)}\cdot 2\cdot\frac{2^{p-1}-1}{p}\cdot pB_{\frac{1}{2}(p-1)} \quad \bmod p.$$

But $pB_{\frac{1}{2}(p-1)} \equiv (-1)^{\frac{1}{2}(p-1)} \bmod p$.[1] Hence

$$T_{p-2} \equiv -2 \cdot \frac{2^{p-1}-1}{p} \mod p.$$

It follows from the above formulas that for no value of k is T_{2k-1} congruent to zero for every odd prime p, though there are, in general, special primes for each value of $k > 3$ for which T_{2k-1} will be congruent to zero.

The residue of T_{p-2} is seen to depend upon the residue of $(2^{p-1} - 1)/p$. The vanishing of this residue has been shown by Wieferich to be a necessary condition in order that Fermat's equation $x^p + y^p = z^p$ shall be solvable in integers prime to p. However, as stated on page 9 of *L'Intermédiaire des Mathématiciens*, January, 1914, this residue does vanish for $p = 1,093$, though for no other primes less than 2,000.

BOOK REVIEWS.

Edited by W. H. Bussey, University of Minnesota.

Plane and Solid Geometry. By Walter Burton Ford, Junior Professor of Mathematics in the University of Michigan, and Charles Ammerman, of the Wm. McKinley High School, St. Louis. Edited by Earle Raymond Hedrick. The Macmillan Company, New York, 1913. ix + 321 + xxxiii pages. \$1.25. Plane and Solid in separate volumes \$0.80 each.

Another interesting text-book of the series edited by Earle Raymond Hedrick, of the University of Missouri, has made its appearance; an attractive volume in a neat brown dress. The work, as we would expect, under Professor Hedrick's editorship, departs from the class of text-books written by scientists not in touch with life. The Euclidean division into books has been abandoned for the modern division into chapters.

The pupil is brought into touch with the subject matter of geometry in the introduction, covering 33 pages, which is divided into three parts: (I) Drawing simple figures, (II) the principal ideas used in geometry, (III) statements for reference. This division of the book contains 111 exercises. Squared paper is used to estimate areas and construct designs.

The arrangement of the subject matter of Plane Geometry in the first five chapters follows that of the usual modern standard text. Logical and original thinking are sure to be developed through the "judicious fusion of theoretical and applied work," here presented. There are approximately 750 exercises in these five chapters. Dr. Frank M. McMurry said, in a paper on "How to Study" given at the Minnesota State Teachers' Association last fall: "Do we in school know how to arrange the work so the pupil will learn to select? Teachers object to neglecting any part of the text. They dwell upon all parts with equal

[1] *Quarterly Journal*, Vol. XXXII, p. 273.

emphasis. Yet in life we have to select, and the exercise of choice is of first importance." The text in question helps the teacher to "exercise choice" very admirably. Different kinds of type are used to indicate the relative importance of theorems. Enough problems are given so that the exceptional pupil may have the stimulus he needs; the medium pupil a fair number; and the slow, plodding pupil enough to make the subject interesting and significant, yet not too difficult. The Appendix to Plane Geometry treats of maxima and minima.

The next three chapters—VI–VIII—are devoted to Solid Geometry. They are made attractive to look at by phantom half-tones. The reviewer would venture the criticism that these "phantom" half-tone engravings, while perhaps making the figures more realistic to the pupil, defeat one of the aims of solid geometry, that is, to develop the imagination. Many of the proofs of the theorems in this division of the book are given in outline only. There are 250 exercises in the Solid Geometry.

A decided innovation in this text is the introduction of three sets of tables. Table I gives the ratios of the sides of right triangles and chords and arcs of a unit circle. This indicates, of course, that the fundamental trigonometric ratios are introduced under proportion and similarity. All starred exercises indicate the use of these ratios. To quote from Article 112, "The relations between chords, arcs, and central angles of the same circle appear vividly in connection with rotations, ... Taking the radius as 1 unit, the lengths of the chords corresponding to various central angles for every degree from 0° to 90° are given in the table of chords." Table II gives powers and roots with an accompanying explanation for use. Table III gives a list of important numbers.

Very few historical notes are given. Such names as Pythagoras, Aristotle, Euclid and Bhaskara are mentioned in connection with famous theorems.

AGNES E. WELLS.

CENTRAL HIGH SCHOOL,
DULUTH, MINN.

A Text-Book of Mathematics and Mechanics, specially arranged for the use of students qualifying for science and technical examinations. By CHARLES A. A. CAPITO. Charles Griffin & Co., London; J. B. Lippincott Co., Philadelphia, 1913. xv + 398 pages.

As the title indicates this is a drill book for the use of persons preparing to take certain examinations; it is admirably adapted to this purpose and one can well believe that it grows out of a long and successful experience in coaching. The theory is presented concisely and, generally, with great clearness. Although extreme brevity characterizes the treatment throughout, space is devoted to the solution of numerous problems, most of which are taken from examination papers of recent date.

The author presumes a knowledge of elementary algebra and trigonometry, and begins with a section on plane analytic geometry, which is followed by sections on differential and integral calculus and mechanics, including hydromechanics

and pneumatics. The section on analytical geometry presents in 46 pages the common theorems relating to the straight line, the circle, and the other conic sections. This is followed by a set of 22 problems with a complete solution for each.

The section on differential calculus covers 59 pages, including 18 pages devoted to the solution of examples. This is brief indeed and quite unsatisfactory from the point of view of its future application in problems of mechanics and physics. In the introduction of the derivative, accuracy of statement is sacrificed to brevity; the process of passing to a limit is not carefully explained, the undetermined form 0/0 is scarcely more than mentioned and that in a confusing manner, and the equation

$$dy = f'(x)dx$$

is said to be a conventional form of the equation

$$\frac{dy}{dx} = f'(x),$$

introduced for the purpose of saving space. Aside from the Taylor and Maclaurin series the applications of the derivative are almost exclusively geometrical. Applications and interpretations of the derivative according to the method of rates, and the uses of the derivative in approximate computations by the method of differentials are practically excluded from consideration. Likewise, in the presentation of the integral calculus the process of integration is not treated as the limit of a process of summation but only as the reverse of the differentiation process. Thus those methods of the calculus which have proved most fruitful in the study of physical phenomena are set aside in favor of those which admit of brief and simple presentation with a minimum of logical difficulty.

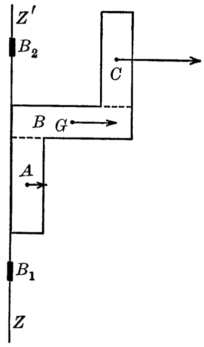
The presentation of Taylor's series is the one which was current in elementary text-books some twenty years ago when the function was assumed expressible in power series and the problem of determining the coefficients was solved by successive differentiations of the series, term by term.

The section on mechanics begins on page 170 and fills the remainder of the 398 pages of the book. The treatment here is in the main admirable, though always very brief. Liberal space (94 pages out of 228) is devoted to the solution of problems. Statics is disposed of in eight pages and kinematical discussions are omitted altogether. The space is therefore practically all devoted to the study of kinetics, with chapters on hydromechanics and pneumatics. The matter of dimension in terms of the fundamental quantities, mass, length and time, is kept well in the foreground. This feature contributes much to the value of the book.

In the words of the author this section "has been written with the intention of avoiding, as far as possible, the unscientific and erroneous expressions still employed by writers of the present day." Just what these expressions are does not appear very prominently, but the author has maintained a satisfactory standard of accuracy and rigor throughout this section. He objects to the use

of the term "centrifugal force," in which objection many of those interested would not concur. This term has been abused by some and confused by others but it has a legitimate place in the theory of mechanics. In defining the unit of time the author states that it is the second, which is 1/86164 of the time the earth takes to complete one revolution about its axis. The second is usually defined as 1/86400 of a mean solar day. It is undoubtedly important that attention should be called to the fact that, owing to its orbital motion, the earth does not complete a revolution about its axis in one mean solar day, but the definition substituted by Capito is open to the objection that it is accurate to five significant figures only, whereas the other is precise as a definition, and capable of expression in terms of other phenomena with an increasing degree of accuracy as the number and accuracy of observations increases.

The unit of mass is defined as ".001 of the mass (at 0° C.) of a standard piece of platinum, etc." The reason for mention of temperature in this connection is not obvious.



An error is committed in the derivation of expressions for the pressure exerted upon bearings when a body of irregular shape rotates about a fixed axis (see p. 326, formula 2). According to this formula the X component of the bearing stress due to the centrifugal action of a rotating mass would be found by assuming the mass to be concentrated at its mass center. That this is incorrect will appear by reference to the figure. The Z-shaped lamina ABC rotates about the shaft ZZ', each of the three sections exerting upon the shaft a force, as indicated by the arrows. Clearly the resultant of these three forces does not pass through G, the center of gravity of the whole mass.

Formally considered the error consists in setting

$$\Sigma mxz = Mr_g\gamma,$$

where r_g is the distance of the center of gravity from the Z axis and γ is the Z coördinate of the center of gravity. This equation is satisfied when the mass M has a plane of symmetry perpendicular to the Z axis. It may be satisfied even when there is no such plane of symmetry, but in general it is not satisfied.[1]

The book has been prepared with great care and attention to detail. Only one misprint was noticed and that was a misspelled word. Numerous figures are furnished and they are clear and helpful. Care has been exercised to maintain a high standard of logical consistency, though some conspicuous lapses have been made in this particular. Perhaps the most serious objection to the book is its failure to develop and utilize those methods of the calculus which have proved most fruitful in the physical sciences. However this criticism would undoubtedly apply with more or less force to most of the elementary calculus texts now in use. The book would not prove satisfactory as a text for use in a first course in any of the branches which it covers, but it would be very helpful in the field for which

[1] See "the theorem of parallel axes" in Routh's *Elementary Rigid Dynamics*, ed. 1905, p. 10.

it was intended, that is, as a review outline for persons preparing to take examinations in elementary mathematics and mechanics.

The plan and style of this book suggest very forcibly some of the advantages and disadvantages of an examination system. Definiteness, conciseness and a certain degree of precision are encouraged but there is danger of stimulating over-use or mis-use of the memory, and discouraging breadth and originality of view and interest in larger problems that cannot be handled adequately within the space of an examination period.

BURT L. NEWKIRK.

PROBLEMS AND SOLUTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

SPECIAL NOTICE. In proposing problems and in preparing solutions, contributors will please follow the form established by the MONTHLY, as indicated on the following pages.

In particular, a solution should be preceded by the number of the problem, the name and address of the proposer, the statement of the problem, and the name and address of the solver.

The solution should then be given with careful attention to legibility, accuracy, brevity without obscurity, paragraphing and spacing, having in mind the form in which it will appear on the printed page.

Please use paper of letter size, write on one side only, leaving ample margins, put one solution only on a single sheet and include only such matter as is intended for publication.

Drawings must be made *clearly* and *accurately* and an extra copy furnished on a *separate sheet* ready for the engraver.

Unless these directions are observed by contributors, solutions must be entirely rewritten by the committee or else rejected.

Selections for this department are made two months in advance of publication.

Please send all solutions direct to the chairman of the committee.

MANAGING EDITOR.

ALGEBRA.

Solutions of 408, 409, 410, 411, 412, 413, 414, and 415 have been received. A solution of 406 is desired.

418. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Form the algebraic equation whose roots are $a_1 = 2\cos(2\pi/15)$, $a_2 = 2\cos(4\pi/15)$, $a_3 = 2\cos(8\pi/15)$, and $a_4 = 2\cos(14\pi/15)$.

419. Proposed by GEORGE A. OSBORN, Massachusetts Institute of Technology.

Show that

$$(1^5 + 2^5 + 3^5 + \cdots + n^5 + 1^7 + 2^7 + 3^7 + \cdots + n^7 = 2(1 + 2 + 3 + \cdots + n)^4.$$

GEOMETRY.

Solutions of 437, 438, 439, 440, 443 have been received. Solutions of 427, 430, 432 and 433 are desired.

447. Proposed by HORACE OLSON, Chicago Illinois.

Given the edge of a regular tetrahedron, find the radius of the circumscribed sphere.

448. Proposed by S. W. REAVES, University of Oklahoma.

Through a given point P within a given angle to draw a line which shall form with the sides of the angle a triangle of a given area [Well's *New Plane Geometry,* (1909), p. 153].

CALCULUS.

Solutions of 352, 354, 355, 356, 357, 358, 359, 361, 362, and 366 have been received. Solutions of 332, 337, 339, 340, 342 are desired.

368. Proposed by PAUL CAPRON, Annapolis, Md.

Develop $\log_{10}(x/\sin x)$ and $\log_{10}(\tan x)/x$, each to three terms, as functions of $\log_{10}\sec x$ and show that if x is less than 7° 15′ then, to five decimals,

$$\log_{10} x = \log_{10}\sin x + 1/3 \log_{10}\sec x = \log_{10}\tan x - 2/3 \log_{10}\sec x.$$

369. Proposed by I. A. BARNETT, Chicago, Ill.

Compute the definite integral $\int_a^b \log x dx$ by direct summation.

MECHANICS.

Solutions of 288, 289, 292, 293, and 294 have been received. Solutions of 268, 269, 274, 275, 277, 278, 279, 286, and 287 are desired.

296. Proposed by C. N. SCHMALL, New York City.

A force F is exerted in moving a horizontal cylinder up an inclined plane by means of a crowbar of length l. If R be the radius of the cylinder, W its weight, ϕ the inclination of the plane to the horizon and ψ the inclination of the crowbar to the horizon, show that

$$F = \frac{WR\sin\phi}{l[1 + \cos(\phi + \psi)]}.$$

297. Proposed by C. N. SCHMALL, New York City.

A shrapnel shell strikes the ground and then explodes, dispersing its fragments in all directions with a common velocity v. If A be the area of the ground covered by the fragments, and if the dimensions of the shell be neglected, show that $A = \pi v^4/g^2$.

NUMBER THEORY.

Solutions of 207, 210, 212, 216, and 218 have been received. Solutions of 189, 191, 192, 196, 200, 205, 208, 209, 211, 213, 214, and 215 are desired.

220. Proposed by E. T. BELL, Seattle, Washington.

Let $[m/n]$ denote the greatest integer that is not greater than m/n; and let the two sets,

$$\left[\frac{m}{m-1}\right];\ \left[\frac{m+1}{m-2}\right];\ \left[\frac{m+2}{m-3}\right];\ \cdots;\ \left[\frac{2m-3}{2}\right],$$

$$\left[\frac{m-1}{m-1}\right];\ \left[\frac{m}{m-2}\right];\ \left[\frac{m+1}{m-3}\right];\ \cdots;\ \left[\frac{2m-4}{2}\right],$$

be denoted by (A) and (B) respectively.

Prove that a necessary and sufficient condition that $2m - 1$ be a prime number is that the excess of the number of even integers in (A) over the number of even integers in (B) shall be equal to the excess of the number of odd integers in (A) over the number of odd integers in (B).

221. Proposed by THOS. E. MASON, Bloomington, Indiana.

Find numbers x such that the sum of the divisors of x is a perfect square [Carmichael, *Theory of Numbers*, p. 17].

SOLUTIONS OF PROBLEMS.

ALGEBRA.

403. Proposed by C. N. SCHMALL, New York City.

A torpedo-boat 40 miles from shore strikes a rock, making a rent in her hull which admits water at the rate of 15 tons in 48 minutes. The ship's pumps can expel 12 tons in an hour. If 60 tons of water is sufficient to sink the boat, find the average rate of steaming so that it may reach the shore just as it is about to sink.

SOLUTION BY CHRISTIAN HORNUNG, Tiffin, O.

If x represents the rate per hour of steaming, then $40/x$ is the number of hours it takes to reach the shore, and in that time $15 \cdot 60 \cdot 40/48x$ tons of water will have entered the hull, and $12 \cdot 40/x$ tons will have been expelled. Hence $(15 \cdot 60/48) \cdot (40/x) - (12/1) \cdot (40/x) = 60$, and $x = 4\frac{1}{2}$.

Also solved by EMMA GIBSON, F. M. MORGAN, HORACE OLSON, CLIFFORD N. MILLS, WALTER C. ELLS, and J. W. CLAWSON.

404. Proposed by V. M. SPUNAR, Chicago, Illinois.

Show that

$$(a+b)(a+b-1)\cdots(a+b-n+1) = a(a-1)(a-2)\cdots(a-n+1)$$
$$+\binom{n}{1}a(a-1)(a-2)\cdots(a-n+2)b+\binom{n}{2}a(a-1)(a-2)$$
$$\cdots(a-n+3)b(b-1)+\cdots+b(b-1)(b-2)\cdots(b-n+1).$$

SOLUTION BY A. M. HARDING, University of Arkansas.

We have

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots;$$

also

$$(1+x)^b = 1 + bx + \frac{b(b-1)}{2!}x^2 + \frac{b(b-1)(b-2)}{3!}x^3 + \cdots.$$

Multiply and equate the coefficients of x^n. Then

$$\binom{a+b}{n} = \binom{a}{n} + \binom{a}{n-1}\binom{b}{1} + \binom{a}{n-2}\binom{b}{2} + \cdots + \binom{b}{n}.$$

If we multiply both members of this equation by $n!$ we obtain the desired result.

Also solved by ELIJAH SWIFT, who proved the proposition by induction.

405. Proposed by E. J. MOULTON, Northwestern University.

Given the alternating series

$$S = 1 - 1/2 + 1/3 - 1/4 + 1/5 - \cdots.$$

(*a*) Let S_n be the sum of the first n terms of the series. Show that in order to make the difference $S - S_n$ numerically less than $1/2k$ (k a positive integer) it is necessary and sufficient to make $n = k$; hence S_{500} differs from S by less than .001. (*b*) Let S_n' be the sum of S_n and 1/2 the $(n + 1)$th term of the series. Show that the difference $S - S_n'$ is numerically less than $1/2n(n+1)$; hence S_{22}' differs from S by less than .001.

Solution by the Proposer.

Denote the numerical value of the remainder after n terms of the series by $|R_n|$.

(*a*) If we compare the two series

$$|R_{n-1}| = \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3} + \cdots$$

$$= \frac{1}{n(n+1)} + \frac{1}{(n+2)(n+3)} + \cdots,$$

$$|R_n| = \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \frac{1}{n+4} + \cdots$$

$$= \frac{1}{(n+1)(n+2)} + \frac{1}{(n+3)(n+4)} + \cdots$$

with the series

$$\frac{1}{2n} = \frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+4} + \frac{1}{n+4} \cdots\right]$$

$$= \frac{1}{n(n+2)} + \frac{1}{(n+2)(n+4)} + \cdots,$$

we have at once $|R_{n-1}| > 1/2n > |R_n|$, from which the statements follow.

(*b*) From the preceding inequality, we see that

$$\frac{1}{2n} > |R_n| > \frac{1}{2(n+1)};$$

and hence, subtracting $1/2(n + 1)$, we have

$$\frac{1}{2n(n+1)} > |R_n| - \frac{1}{2(n+1)} > 0.$$

Letting S be the value of the series, we see

$$S - S_n' = (-1)^n \left[|R_n| - \frac{1}{2(n+1)}\right]$$

and the statements (*b*) follow.

Remark. Perhaps the chief interest of the problem comes from a comparison with a conclusion obtainable immediately from a well-known general theorem on alternating series (Osgood, *Introduction to Infinite Series*, p. 13), namely, that, for our series, to make $|R_n| < .001$ it is *suf-*

ficient to take $n = 1000$; we see that a much smaller number, 500, is also sufficient (and necessary). Part (*b*) of the problem was originally suggested to the proposer by the fact that when the values of $S_1, S_2, S_3, \cdots$ are plotted on a line, two successive points lie on opposite sides of S and about equally distant from S. The midpoint between S_n and S_{n+1} is S_n'. The value S_n' as an approximation to S may also be obtained as follows: We may write

$$S = S_n + (-1)^n\left[\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \cdots\right]$$

$$= S_n + \frac{(-1)^n}{n+1}\left[\frac{1}{(n+1)(n+2)} + \frac{1}{(n+3)(n+4)} + \cdots\right]$$

$$= S_n + \frac{(-1)^n}{n+1} - (-1)^n\left[\frac{1}{(n+2)(n+3)} + \frac{1}{(n+4)(n+5)} + \cdots\right].$$

Adding the last two equations and dividing by 2, we have

$$S = S_n' + (-1)^n\left[\frac{1}{(n+1)(n+2)(n+3)} + \frac{1}{(n+3)(n+4)(n+5)} + \cdots\right].$$

If we compare this with the second equation we find

$$|S - S_n'| < \frac{1}{n+3}\cdot|S - S_n| \quad \text{or} \quad |S - S_n'| < \frac{1}{2n(n+3)},$$

which is a slightly better test than that given in the problem. In particular it follows that $|S - S_{21}'| < .001$.

407. Proposed by E. B. ESCOTT, University of Michigan.

In computing the values of the natural logarithms of 2, 3, and 5 by the following formulas:

$$\log 2 = 2(7P + 5Q + 3R),$$

$$\log 3 = 2(11P + 8Q + 5R),$$

$$\log 5 = 2(16P + 12Q + 7R),$$

where P, Q, and R are numbers which were computed by infinite series (G. Chrystal, Algebra, Part II, chapt. 28), it is found, on comparing the results with the known values of these logarithms to 15 decimals, that there are the following errors: -2533, -4052, and 6080, respectively. Find the errors in P, Q and R.

SOLUTION BY S. A. JOFFE, New York City.

Denoting the computed values of P, Q and R by the same capital letters with primes, and the errors of computation by the corresponding small letters, we will have: $P' = P + p$; $Q' = Q + q$; $R' = R + r$.

Since $\log 2 = 2(7P + 5Q + 3R) = 2[7(P' - p) + 5(Q' - q) + 3(R' - r)]$, and the computed value $\log' 2 = 2(7P' + 5Q' + 3R')$, we see that the error in $\log' 2$, when compared with the known value of $\log 2$, is $\log' 2 - \log 2$, or

$$2(7p + 5q + 3r) = -2533; \tag{1}$$

similarly:

$$2(11p + 8q + 5r) = -4052, \tag{2}$$

$$2(16p + 12q + 7r) = 6080. \tag{3}$$

This system of simultaneous equations may be solved as follows:

Multiplying (1), (2), and (3) by 4, -1, and -1 respectively, we eliminate q and r, and obtain $2p = -4 \times 2533 + 4052 - 6080$, or $p = -6080$.

Multiplying (1), (2), and (3) by -3, -1, and 2 respectively, we eliminate r and p, and obtain $2q = 3 \times 2533 + 4052 + 2 \times 6080$, or $q = 11{,}905\frac{1}{2}$.

Multiplying (1), (2), and (3) by -4, 4, and -1, respectively, we eliminate p and q, and obtain $2r = 4 \times 2533 - 4 \times 4052 - 6080$, or $r = -6078$.

Hence the errors in computing P, Q, and R are respectively -6080; 11905.5 and -6078.

Excellent solutions were also received from WALTER C. EELLS, J. W. CLAWSON, and the PROPOSER.

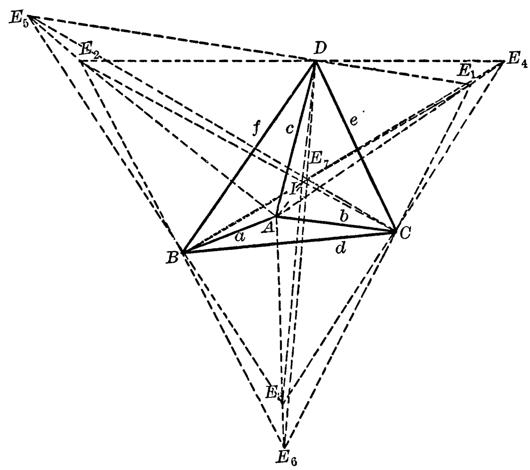
GEOMETRY.

417. Proposed by R. P. BAKER, University of Iowa.

Enumerate the points in which the twelve dihedral bisector planes of a tetrahedron meet, find their multiplicity and account for the 220 points which 12 planes in general determine.

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

Call the planes bisecting the dihedral angles internally by the letters a, b, c, d, e, f; the planes perpendicular to these which bisect the dihedral angles externally a', b', c', d', e', f'.



Six of these planes pass through the vertex A. Six planes in general determine 20 points. Now a, b, c intersect in the straight line joining A to the center of the inscribed sphere and the opposite ex-center; a', b, c' intersect in a line through A and two of the centers of escribed spheres; a, b', c' in a line joining A to another pair of ex-centers; a', b', c in another such line. These 20 points then reduce to a vertex, taken 16 times, and 4 straight lines passing through that vertex. Since there are four vertices, we can thus account for 80 of the 220 points.

Six of the 12 planes pass through the center of the inscribed sphere. Six planes in general determine 20 points. However a, b, c intersect in the straight line AI; and similarly the three planes passing through each of the other three vertices determine 3 lines. This eliminates 4 of the 20 points from consideration,

since these four lines have already been considered in the paragraph above. Thus the planes intersect three by three at I, taken 16 times. Similarly each of the seven ex-centers is determined 16 times. For instance a', b', f', c, d, e' intersect at E_6. These six planes determine the point E_6 16 times and four lines AE_6, BE_6, CE_6, DE_6, already considered above. This gives us then 8 points each repeated 16 times or 128 more of the 220 points.

Twelve points remain to be located. Now e intersects a and a' at a point on AB equidistant from two faces of the tetrahedron. And e' intersects a and a' at a second point on AB equidistant from the same two faces. Similarly there are two points on each of the six edges in which three planes intersect. This gives the remaining 12 points, each determined once.

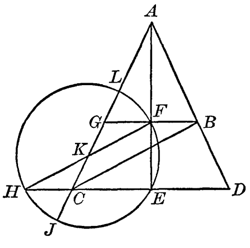
The 220 points, then, are four vertices, each 16 times; an in-center and 7 ex-centers each 16 times; 2 points on each of the 6 edges; and four straight lines from each of the four vertices to the in-center and ex-centers, these centers lying in pairs on these lines, 16 lines in all. Total 204 points and 16 lines. The figure shows these points and lines, except the 12 points, 2 on each edge.

434. Proposed by CLIFFORD N. MILLS, Bloomington, Ind.

ABC is any triangle with sides, a, b, c. Prove by purely geometrical methods that the area $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$.

SOLUTION BY E. E. WHITFORD, New York City.

Lay off $AG = AB$, draw GB, draw $CD \| GB$ meeting AB prolonged at D. Through the middle point F of GB draw AE meeting CD in E. Draw FH through K, the mid-point of GC, meeting CE prolonged at H. With K as center and KF as radius describe circle. AE is $\perp$ to GB and CD, $KF \|$ to CB and $HF = CB$. H and E are on the circumference.



$$\triangle ABC = \triangle ADC - \triangle BDC = CE \cdot AE - CE \cdot EF = CE \cdot AF.$$

By similar $\triangle$'s, $CE : AE = GF : AF$. Hence $CE \cdot AF = AE \cdot GF = AE \cdot HC$,

$$\triangle ABC = \sqrt{CE \cdot AF}\sqrt{CE \cdot AF} = \sqrt{CE \cdot AF}\sqrt{AE \cdot HC},$$

$$\triangle ABC = \sqrt{CE \cdot AF \cdot AE \cdot HC} = \sqrt{(AF \cdot AE) \cdot (CE \cdot HC)},$$

$$\triangle ABC = \sqrt{(AJ \cdot AL) \cdot (CL \cdot CJ)} = \sqrt{s(s-c)(s-a)(s-b)}.$$

For $AJ = AK + KJ = \frac{1}{2}(AC + AG) + KF = \frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}a = s$, $AL = AJ - JL = s - a$, $CL = GJ = AJ - AG = s - c$, $CJ = AJ - AC = s - b$.

This solution is based on Legendre.

Also solved by J. L. RILEY.

435. Proposed by R. M. MATHEWS, Riverside, Calif.

From a fixed point A perpendiculars are dropped to the tangents drawn to a circle whose center is O. Prove that the locus of the feet of the perpendiculars is a limaçon.

SOLUTION BY WALTER C. EELLS, U. S. Naval Academy.

Let a be the radius of the circle, and let distance OA be c. Take x-axis through O and A and y-axis through $O \perp OA$. Then the equation of the circle is $x^2 + y^2 = a^2$, and that of all tangents to it with the slope m is

$$y = mx \pm a\sqrt{1 + m^2}. \tag{1}$$

The equation of all lines through $A \perp$ to the system of lines given by (1) is

$$y = -\frac{1}{m}(x - c). \tag{2}$$

Transforming to new origin at A by equations $x = x' + c$, $y = y'$ and dropping primes, equations (1) and (2) become

$$y = m(x + c) \pm a\sqrt{1 + m^2}, \tag{3}$$

$$m = -\frac{1}{y}x. \tag{4}$$

Eliminating the parameter m by substituting (4) in (3), clearing of fractions, transposing and squaring, we have the well-known equation

$$(x^2 + y^2 + cx)^2 = a^2(x^2 + y^2),$$

representing the three standard forms of the limaçon, according as $c <, =, >, a$ i. e., according as A is within, on, or without the circle.

Also solved by J. W. CLAWSON, S. W. REAVES, ELIJAH SWIFT, A. M. HARDING, DANIEL KRETH, C. N. SCHMALL, HORACE OLSON, and the PROPOSER.

CALCULUS.

Additional solutions were received too late for the June issue from J. L. RILEY for 335, and from A. M. HARDING and J. W. CLAWSON for 345.

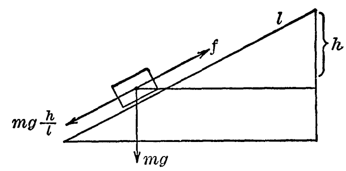
346. Proposed by C. N. SCHMALL, New York City.

Give the height of an inclined plane, to determine its length so that a given force acting on a given mass parallel to the plane may draw it up in the shortest time.

Solution by B. L. Libby, Ann Arbor, Mich.

Using the notations indicated in the figure, we have the resultant force acting upwards

$$f - mg \cdot \frac{h}{l}.$$



The acceleration required is therefore given by

$$a = \frac{fl - mgh}{ml}$$

Hence $dl/dt = (fl - mgh/ml)t$, and $l = (fl - mgh/2ml)t^2$, from which

$$t^2 = 2ml^2/(fl - mgh).$$

Differentiating, we have

$$t\frac{dt}{dl} = \frac{mfl^2 - 2m^2ghl}{(fl - mgh)^2}.$$

The usual test for a minimum for t gives

$$l = \frac{2mgh}{f} \quad \text{and} \quad t = 2m\sqrt{\frac{2gh}{f}}.$$

Also solved by Horace Olson and P. Peñalver.

347. Proposed by R. D. Carmichael, Indiana Univ.

Show that the differential equation

$$9\left(\frac{d^2y}{dx^2}\right)^2\frac{d^5y}{dx^5} - 45\frac{d^2y}{dx^2}\frac{d^3y}{dx^3}\frac{d^4y}{dx^4} + 40\left(\frac{d^3y}{dx^3}\right)^3 = 0$$

remains unchanged when the variables x and y undergo any projective transformation.

Solution by A. M. Harding, University of Ark.

Any projective transformation can be reduced to a succession of integral transformations of the form

$$x = aX + bY + c, \qquad y = a_1X + b_1Y + c_1, \tag{1}$$

combined with the particular transformation

$$x = \frac{1}{X}, \qquad y = \frac{Y}{X}.$$

Hence it will be sufficient to show that the given equation remains invariant under each of these transformations. From (1) we find

$$y' = \frac{a_1 + b_1Y'}{a + bY'}, \; y'' = \frac{(ab_1 - a_1b)Y''}{(a + bY')^3}, \; y''' = \frac{(ab_1 - a_1b)[(a + bY')Y''' - 3b(Y'')^2]}{(a + bY')^5},$$

$$y^{\text{iv}} = \frac{(ab_1 - a_1b)[(a + bY')^2Y^{\text{iv}} - 10b(a + bY')Y''Y''' + 15b^2(Y'')^3]}{(a + bY')^7},$$

$$y^{\text{v}} = \frac{(ab_1 - a_1b)[(a + bY')^3Y^{\text{v}} - 15b(a + bY')^2Y''Y^{\text{iv}} - 10b(a+bY')^2(Y''')^2 + 105(a + bY')b^2(Y'')^2Y''' - 105b^3(Y'')^3]}{(a + bY')^9}$$

From (2) we find

$$y' = Y - XY', \quad y'' = X^3Y'', \quad y''' = -X^4(3Y'' + XY'''),$$

$$y^{\text{iv}} = X^5[12Y'' + 8XY''' + X^2Y^{\text{iv}}]$$

$$y^{\text{v}} = -X^6[60Y'' + 60XY''' + 15X^2Y^{\text{iv}} + X^3Y^{\text{v}}].$$

It can be easily shown by substitution that each transformation leaves the given equation invariant.

Also solved by J. W. Clawson and Geo. W. Hartwell.

349. Proposed by C. N. SCHMALL, New York City.

If $y = a \cos (\log x) + b \sin (\log x)$, eliminate the constants a and b and obtain the equation

$$\frac{x^2d^2y}{dx^2} + \frac{xdy}{dx} + y = 0.$$

Solution by C. C. Steck, New Hampshire State College.

By differentiating the equation $y = a \cos (\log x) + b \sin (\log x)$, we obtain

$$\frac{xdy}{dx} = -a \sin (\log x) + b \cos (\log x).$$

Differentiating this and multiplying the resulting equation by x, we get

$$\frac{x^2d^2y}{dx^2} + \frac{xdy}{dx} = -a \cos (\log x) - b \sin (\log x) = -y.$$

From the last two equations readily follows the desired result

$$\frac{x^2d^2y}{dx^2} + \frac{xdy}{dx} + y = 0.$$

Also solved by M. E. Graber, J. B. Smith, J. W. Clawson, P. Peñalver, C. Hornung, Elmer Schuyler, A. M. Harding, W. W. Beman, A. L. McCarty, H. L. Slobin, Clifford N. Mills, Francis Rust, I. A. Barnett, F. C. Reisler, G. W. Hartwell, S. W. Reaves, Richard Morris, Albert R. Nauer, Barnem Libby, and Walter C. Eells.

351. Proposed by C. N. SCHMALL, New York City.

In the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ are given e the eccentricity, and the angle φ which the normal at any point P (on the curve) makes with the major axis. If R is the radius of curvature at P prove that

$$R = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \varphi)^{\frac{3}{2}}}.$$

SOLUTION BY GEO. W. HARTWELL, Hamline University, St. Paul, Minn.

For the given ellipse

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}, \tag{1}$$

$$\frac{d^2y}{dx^2}=-\frac{b^4}{a^3y}. \tag{2}$$

Also

$$\frac{dy}{dx}=\cot\varphi. \tag{3}$$

Making these substitutions in the usual formula for radius of curvature,

$$R=\frac{(1+\cot^2\varphi)^{3/2}}{\dfrac{b^4}{a^2y^3}}=\frac{a^2y^3}{b^4\sin^3\varphi}. \tag{4}$$

From (1) and (3)

$$b^2x=-a^2y\cot\varphi. \tag{5}$$

Substituting this value of x in the equation of the ellipse, we have

$$a^4y^2\cot^2\varphi+b^2a^2y^2=a^2b^4.$$

Hence,

$$y^2=\frac{b^4}{b^2+a^2\cot^2\varphi}=\frac{b^4\sin^2\varphi}{a^2\cos^2\varphi+b^2\sin^2\varphi}=\frac{b^4\sin^2\varphi}{a^2-(a^2-b^2)\sin^2\varphi}. \tag{6}$$

But $a^2-b^2=a^2e^2$. Hence

$$y^3=\frac{b^6\sin^3\varphi}{a^3(1-e^2\sin^2\varphi)^{3/2}}$$

and

$$R=\frac{b^2}{a(1-e^2\sin^2\varphi)^{3/2}}.$$

But $b^2=a^2(1-e^2)$. Hence,

$$R=\frac{a(1-e^2)}{(1-e^2\sin^2\varphi)^{3/2}}.$$

Also solved by RICHARD MORRIS, C. C. STECK, F. M. MORGAN, J. W. CLAWSON, S. W. REAVES, and J. G. GONZALES.

MECHANICS.

A solution of 273 was received from J. W. CLAWSON too late for the June issue.

271. Proposed by B. F. FINKEL, Springfield, Mo.

A hollow spherical shell is filled with a frictionless fluid and rolls down a rough inclined plane. After rolling t seconds the fluid suddenly solidifies. Determine the subsequent motion of the spherical shell.

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

Let m be the mass of contained fluid; m' the mass of the shell alone; α the angle of inclination of the plane; a the radius of sphere; x the distance the sphere

has rolled from rest after a time T; θ the angle turned through by the sphere after time T; I the moment of inertia of the rotating mass; and T' the time that has elapsed after solidification.

Using the formula for torque, we have

$$(m + m')ga \sin \alpha - (m + m')\ddot{x}a = I\ddot{\theta}.$$

But $x = a\theta$. Hence, $\ddot{x} = a\ddot{\theta}$. Hence, $(m + m')ga^2 \sin \alpha = (m + m')a^2\ddot{x} + I\ddot{x}$, from which we find

$$\ddot{x} = \frac{(m + m')ga^2 \sin \alpha}{(m + m')a^2 + I}.$$

Now, for the first t seconds, $I = 2/3 \cdot m'a^2$, since the fluid does not rotate. (The shell is assumed to be of so small a thickness that its square may be neglected in finding I.) Hence

$$\ddot{x} = \frac{(m + m')gs \text{ in } \alpha}{(m + m') + 2/3 \cdot m'}, \quad \dot{x} = \frac{3(m + m')g \sin \alpha}{3m + 5m'} \cdot t, \quad x = \frac{3(m + m')g \sin \alpha}{3m + 5m'} \cdot \frac{t^2}{2}.$$

After the first t seconds, $I = 2/3 \cdot m'a^2 + 2/5 \cdot ma^2$.

Hence,

$$\ddot{x} = \frac{(m + m')g \sin \alpha}{7/5 \cdot m + 5/3 \cdot m'}, \quad \dot{x} = \frac{15(m + m')g \sin \alpha}{21\, m + 25m'} T' + \frac{3(m + m')g \sin \alpha}{3m + 5m'} \cdot t,$$

and

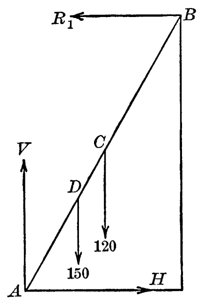
$$x = \frac{15(m + m')g \sin \alpha}{21\, m + 25\, m'} \frac{T'^2}{2} + \frac{3(m + m')g \sin \alpha}{3m + 5m'} T' \cdot t + \frac{3(m + m')g \sin \alpha}{3m + 5m'} \cdot \frac{t^2}{2}.$$

285. Proposed by RICHARD P. LOCHNER, Philadelphia, Pa.

A ladder 25 ft. long, weighing 120 lbs., leans against a vertical wall. Its foot is prevented from slipping on the plane by a peg driven into the ground 7 feet from the wall. If a man weighing 150 lbs. is one-third the way up the ladder, what is the reaction on the peg, the ground, and the wall?

SOLUTION BY RICHARD MORRIS, Rutgers College.

Denote the reaction against the vertical wall by R_1, that against the peg by H and that against the ground by V, the weight of the ladder being concentrated at its mid-point.



Taking moments about A, we get

$$24\, R_1 = \frac{7}{3} \cdot 150 + \frac{7}{2} \cdot 120 = 770.$$

Hence $R_1 = 770/24 = 32\,{}^1/_{12}$ lbs. $R_1 = H$, since horizontal forces are equal. Also $V = 120 + 150 = 270$, resolving vertically.

Also solved by S. W. REAVES, J. B. SMITH, CIFFORD N. MILLS, A. M. HARDING, and J. W. CLAWSON.

NUMBER THEORY.

204. Proposed by E. T. BELL, New York City.

Show that a necessary and sufficient condition that $6n + 1$ be a prime number is that no one of the quantities $(3m - r)/(2r + 1)$ for $r = 1, 2, 3, \cdots n - 1$ be an integer. Similarly for $6n - 1$, the quantities being $(3n - r)/(2r - 1)$ for $r = 2, 3, 4, \cdots n$.

SOLUTION BY ELIJAH SWIFT, Princeton, N. J.

Suppose first that the condition be not satisfied. Then we have for some $r < n$, $(3n - r)/(2r + 1) =$ an integer, say α, which is equivalent to the equation $6n + 1 = (2r + 1)2\alpha + 2r + 1$. Hence, $6n + 1$ has the factor $2r + 1$ and is not prime, so that the condition is necessary.

Again suppose that $6n + 1$ is not prime. We shall see that the given condition is not satisfied. In this case the odd number $6n + 1$ has two odd factors, which we will call $2r + 1$ and $2\alpha + 1$. Each must be as large as 5 since 3 is evidently not a factor of $6n + 1$. If $2\alpha + 1 \geqq 5$, $2r + 1 \leqq (6n + 1)/5 = n + (n + 1)/5$. But this is less than $2n - 1$ if $n \geqq 2$, and if $n = 1$, $6n + 1$ is prime. Hence $r \leqq n - 1$. Reversing the algebraic work above we infer from $6n + 1 = (2r + 1)(2\alpha + 1)$ the equation $(3n - r)/(2r + 1) = \alpha$. Hence the condition is sufficient.

The proof of the second half of the theorem is similar mutatis mutandis.

206. Proposed by R. D. CARMICHAEL, Indiana University.

Prove that the sum of the sixth powers of two integers cannot be the square of an integer.

SOLUTION BY ELIJAH SWIFT, Princeton, N. J.

We are required to show that the equation $a^6 + b^6 = c^2$, where a, b, c are integers, is impossible. We may suppose that a, b, and c are prime to each other, as any common factor may be divided out. Now all integral solutions of the equation $x^2 + y^2 = z^2$, where x, y, z are prime to each other are given by the formulas $x = 2mn$, $y = m^2 - n^2$, $z = m^2 + n^2$, where m and n are integers prime to each other, and one is even and one odd. We must, then, have $a^3 = 2mn$, $b^3 = m^2 - n^2$. Suppose that m is even and n odd. Since $2mn$ is a cube, and $2m$ and n are prime to each other, each must be a cube, and $2m$ must contain the factor 8, and m the factor 4. Call $2m = 8\alpha^3$, $n = \beta^3$. Substituting these values in the formula for b^3, we have $b^3 = 16\alpha^6 - \beta^6 = (4\alpha^3 - \beta^3)(4\alpha^3 + \beta^3)$. Since $2m$ and n have no common factor, $4\alpha^3$ and β^3 have none, and $4\alpha^3 - \beta^3$ and $4\alpha^3 + \beta^3$ must also be prime to each other. Hence each is a perfect cube, since their product is, and we have $4\alpha^3 + \beta^3 = q^3$, $4\alpha^3 - \beta^3 = p^3$.

But from these two equations follows at once by addition $8\alpha^3 = p^3 + q^3$ or $r^3 = p^3 + q^3$, which is impossible. If n is even and m is odd the same method applies.

This method is general and may be applied to prove that if a, b, c, and n are positive integers (A) $a^{2n} + b^{2n} = c^2$ is possible only if there exist positive integers p, q, and r such that (B) $p^n + q^n = r^n$. If we assume the truth of Fermat's

theorem—namely that (B) is impossible if $n > 2$ — we have shown the impossibility of (A) for every value of n except $n = 1$, or $n = 2$. For $n = 1$ (A) is of course possible; its impossibility for $n = 2$ may be shown by this same method. See Bachmann, *Niedere Zahlentheorie,* vol. II, p. 453.

MISCELLANEOUS QUESTIONS.

EDITED BY R. D. CARMICHAEL.

QUESTIONS.

13. What are the most important of the newer problems that confront teachers of high school mathematics, and why are they important?

REPLY.

9. What is the present state of experience with coördinated courses in high school mathematics? What contribution does this promise to the development of mathematics teaching in high schools? What about the corresponding matters in college mathematics?

I. REMARKS BY EDITH LONG, Lincoln, Nebraska.

That there has been and still is an unrest among thoughtful high school teachers of mathematics is manifest. This problem certainly exists: How are we to adapt the courses in algebra and geometry to the experiences of high school students, in order that these may contribute to their broader development in both a cultural and a practical way? On the one hand we have the student who is preparing for the university and on the other hand the student whose school life ends with the high school or who, going on to the university, will not continue the subject of mathematics. The one who would solve this problem must have these two classes in view, and the solution for the one must be the solution for the other since it is only in the largest schools that distinction can be made by having separate classes, even if it were advisable. It is for the high school teacher to work out a simple course which will serve the purpose of college preparation as well as the one now followed and which at the same time will give to the boy or the girl who does not go on a clear and simple notion of the nature and possibilities of mathematics as a whole. It ought to be possible for pupils to leave the high school with other impressions than that this great subject is embodied in two narrow and distinct divisions called algebra, where one works exercises, and geometry, where one demonstrates theorems. Some impressions of tangible and present value should be made to replace the vague feeling of subtle virtue, not to be understood now, but which in future years it is hoped will show its value and repay for long and weary hours spent.

There is no conflict between these two aims. A course that will satisfy the latter purpose will serve the former far better than those we now have. But before such a course can be formulated and put into effect the barriers of prejudice must be broken down. The tandem plan of algebra, geometry, trigonometry, analytical geometry, and calculus, running through the high school, and the first two years of college, should give way to a new course which will unify these sub-

jects. The simple parts with concrete basis should be given in the high school and from this on in one continuous course the subject should be developed in the college. It is, indeed, strange that this plan has been reversed and that much of the clear, simple mathematical intuitions are postponed to the college while much abstract work is required of the ninth-grade, thirteen-year-old child; for example, in connection with the theory of exponents and radicals and the binomial theorem, which apart from the claims of formal discipline, can be of no use to any one until they are applied in the higher mathematics. But any child of average brain power can understand and enjoy every phase of either algebra or geometry which is developed from measurement and drawing; and in this are to be included the elementary notions of analytical geometry and trigonometry which, to avoid paralyzing names, should come under the head of graphic work.

He who runs may read the handwriting on the wall. Either we must adapt the course to the student or it will be swept entirely from the curriculum in the frenzied demand for humanized and vocational courses of study. Viewed purely from the standpoint of preparation for college, the great slaughter of students at the end of the first semester points to the weakness of the present plan. How could it be otherwise under an arrangement which is so contrary to the principles of pedagogy and child-psychology? First comes algebra, written in a language entirely strange to the child, with rules to be memorized and used in drill work for rapidity and accuracy. Now algebra is held in the schools under the supposition that it gives reasoning power, but such memory work does not stimulate thought power. And of what use is accuracy if there is no thought back of it? It is accuracy of thought primarily and not merely accuracy of execution that should be sought in mathematics. If drill in mechanical accuracy and rapidity are the aim, a course in business arithmetic would answer the purpose better.

After a year of drill, when the student is beginning to acquire some efficiency in this line, another extreme kind of mental activity is introduced. Geometry, as it is taught, demands intensive deductive reasoning in place of rule and drill. This is continued for a year when the child is suddenly switched back to his drill work for another half year. Then it is all dropped until a year and half later the student is called upon, not to continue his algebra and geometry in the way in which he has been taught, but to apply them in ways in which he has not been taught. It is not strange that so many fail in the first year of college mathematics.

From the standpoint of the course finished in the high school, and this is really the more important, there can be no doubt that readjustment must take place if the student is to reap the best results. To be successful this readjusted course must be built upon the experiences of the child in connection with the measurement of lines, areas and solids, first brought into use by drawings, so as to appeal to the eye, and then by expression in algebraic symbols. Thought conveyed to the brain through the eye is lasting. The picture appeals and lingers. There is little of algebra which cannot be based on some picture and none that should be required of ninth-grade students. As far as possible the picture thought

should be constantly seen in the algebraic sentence. No sentence without thought should be demanded in algebra as it is in other studies. Thought to the child means concrete sight thought. It is not only a child trait but a universal trait that the thought which the eye conveys is most effectively conceived. Data expressed in a list of figures make little impression, while a graph from the same list is convincing. A well-drawn cartoon may do more damage to a cause than a season of speeches. The time spent in drill may be lessened by one half if in the algebraic symbols is seen a shorthand writing of an English sentence, and at the same time the student will acquire a steady thoughtful rate of manipulation.

For example, why is the thermometer the best illustration of negative quantities? Because in the thermometer we have a perfect mechanical graph. It has the scale line, an arbitrarily chosen zero, an arbitrary unit of measure, and a real little point running up and down the scale, adding both positive and negative quantities. Children delight in working with it. The picture of this is the graph and the student can always draw this picture. But the picture has broader possibilities. It can be made to illustrate quantity of every kind that admits of the positive and negative idea.

But it is not for the sake of training the student to be more expert in the solution of exercises that I make the plea for the concrete treatment of algebra. It is for the sake of giving as a basis of the entire development something tangible, something definite upon which the student can build the work for himself. Having in hand material which he understands, he should be able to organize his work and develop the subject with some independence. It should not be necessary for the teacher to take the lead every time a new phase of the subject comes up. With a clear notion that factoring has its counterpart in finding the dimensions of rectangles whose areas are given, and previously having studied the nature of the rectangle, the student should by drawing find his own way to factoring, first arithmetic numbers, and then simple quadratic expressions. The teacher should guide, not develop, the work. To always carefully develop a subject before assigning new work may do more harm than to allow a class sometimes to flounder, for in the latter case there are always some who will find their way out with great strength, while in the former case every member of the class is crippled. With material in hand and lessons carefully assigned there will be no floundering, not even when the student passes from algebra to geometry. A student who has developed his own algebra from geometric figures scarcely realizes that there is any change in subject. He has already learned to take the initiative in organizing his work and he is not handicapped. As the geometry progresses he can work out the simpler notions of trigonometric ratios and thus widen the field of application to real things about him.

In this reconstruction we need the coöperation of the university teacher. While it is true that many school superintendents declare that they pay no attention to what the university demands, nevertheless, the present arrangement of the mathematical course in the high school was originally made through the influence of college entrance requirements and nothing would facilitate the readjustment

now so much as a liberal attitude on the part of the colleges and universities in accepting the unified course for admission. There is no line along which the child-study investigators could better direct their attention than to placing mathematics on a sounder pedagogical basis; for there is no subject which causes such waste of time and such humiliation through failure on the part of the child.

NOTES AND NEWS.

EDITED BY W. DEW. CAIRNS.

An article (in German) by Professor T. H. Gronwall of Princeton University on the summability of Laplace and Legendre series appears in the June number of *Mathematische Annalen.*

Professor C. J. Keyser discourses upon " The Study of Mathematics " in his inimitable style in the *Columbia University Quarterly* for June.

Mr. W. W. Kustermann has been appointed to an instructorship in mathematics in the University of Michigan, to succeed Mr. E. B. Escott who resigned recently.

Professor R. C. Archibald has been added to the editorial board of the *Bulletin* of the American Mathematical Society.

At Purdue University Mr. Ralph B. Stone has been promoted to an assistant professorship of mathematics; Dr. Thomas E. Mason, Mr. O. W. Albert and Mr. Charles K. Robbins have been appointed to instructorships; Professor William Marshall has been given leave of absence for a year of study in France.

The June issue of *School Science and Mathematics* contains a paper by Professor W. B. Ford of the University of Michigan on " The Future of Geometry " read before the spring meeting of the Michigan Schoolmaster's Club; and a paper by Miss I. E. Holroyd entitled " Mathematics in the Education of Girls " read before the Mathematical Club of the Kansas State Agricultural College.

Mr. W. A. Reinert, of the Oak Park, Ill., high school, has been appointed to an instructorship in mathematics at the Michigan Agricultural College, Lansing, Mich. He is a bachelor of the University of Wisconsin and has been taking graduate work at the University of Chicago.

Professor David Eugene Smith of Teachers College, Columbia University, has leave of absence for the next academic year. He attended the Napier tercentenary at Edinburgh where he made an address, and also the Roger Bacon celebration in Oxford, where he read a paper on " Roger Bacon as a mathematician."

The Mathematics Teacher for June prints an interesting address by Dean F. C. Ferry of Williams College on " Mathematics: The Subject and the Teacher."

Mr. M. G. Gaba, who has just taken the doctor's degree at the University of Chicago, has been appointed to an instructorship in mathematics in the Carnegie Institute of Technology at Pittsburg, Pa.

Professor F. B. Wiley, who has been on leave of absence from Denison University, Granville, Ohio, for the purpose of graduate study, took the doctor's degree in mathematics at the University of Chicago at the convocation held in August.

Science, May 8, 1914, contains a short note on the first meeting and the organization of the " Committeee of One Hundred on Scientific Research," which was recently appointed by the American Association for the Advancement of Science. Ten members of this committe are connected with bureaus of the federal government. The educational institutions which are represented on this committee by at least three men each are as follows: Harvard (12), Chicago (9), Columbia (7), Yale (5), Johns Hopkins (5), Carnegie Institution (5), Massachusetts Institute of Technology (4), Cornell (3), Illinois (3), and Stanford (3). Charles W. Eliot, president emeritus of Harvard University, is chairman of the committee, which will hold its next meeting at Philadelphia on December 28, 1914. The committee aims to consider broad questions relating to research " under the government, in the universities and in other institutions."

A Heft of the *Encyklopädie der Mathematischen Wissenschaften*, devoted to elementary geometry, appeared in June of the present year. It discusses many questions which are of interest to the teachers of elementary geometry in the high schools. The parts of this encyclopedia which have been published contain about ten thousand pages. Several additional parts are in press.

The May, 1914, number of the *Revista de la Sociedad Matemática Española* contains a brief account of the first congress of mathematical philosophy which was held at Paris during last April.

An interesting article on the teaching of mathematics in Japan appeared in the May number of the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht.*

The mathematics section of the California High School Teachers Association met at the University of California July 2 and 3, 1914. Among the papers presented were the following: " How shall the isolated teacher of mathematics keep up his interest? " by Professor D. N. Lehner, University of California; " Mathematics and life in the intermediate school," by Miss Thirmuthis Brookman, special instructor in the University of California summer session; " The

university and the secondary schools," by Dr. Henry W. Stager, of Fresno Junior College, and chairman of the section. A fuller report of this meeting will appear in a later issue of the MONTHLY.

At the University of Minnesota W. H. Bussey has been promoted to an associate professorship in mathematics, and H. L. Slobin and W. F. Holman have been promoted to assistant professorships.

The Macmillan Company of London has recently published "A first book of geometry" by J. V. H. Coates as one of the series entitled "First Books of Science." It contains only 142 pages, and sells for 50 cents.

Isis, the new journal devoted to the general history of science, will in the future publish an *edition de luxe* on thin paper for the convenience of bibliographers and librarians.

In connection with the commemoration exercises of the seventh centenary of Roger Bacon's birth a volume of essays on various aspects of his activity has been published by the Oxford University Press. Professor David Eugene Smith is one of the contributors. His paper is entitled "The place of Roger Bacon in the history of mathematics." Bacon's works on optics, mirrors, physics, alchemy and chemistry, and medicine are also treated in special essays. The volume is edited by A. G. Little; the other contributors are L. Baur, F. Picavet, Cardinal Gasquet, S. A. Hirsch, Eilhard Wiedemann, S. Vogl, J. Wurmschmidt, Pierre Duhem, M. M. P. Muir, H. W. L. Hime, E. Withington, and J. E. Sandys.

Columbia University will commemorate Bacon's birth by appropriate exercises at the opening of the college year this fall. A pageant is planned as part of the program. The Research Club of the University of Michigan devoted the April meeting to addresses in honor of Bacon. Professors Dow, Tatlock, Lloyd, and Guthe contributed the papers. Professor Guthe's paper on "Roger Bacon as a scientist" appeared in the August issue of *The Open Court Magazine*.

Volney H. Wells, A.B. Olivet College, graduate student 1912–1914 at the University of Michigan, has been appointed instructor in mathematics at the University of Michigan.

Professor Elijah Swift, of Princeton University, has been appointed head of the department of mathematics at the University of Vermont.

Miss Susan Rose Benedict, fellow in mathematics at the University of Michigan in 1913–1914, has been promoted to an associate professorship of mathematics at Smith College. Miss Benedict received the Ph.D. in June at the University of Michigan. Her thesis is entitled, "A comparative study of the early treatises which introduced into Europe the Hindu art of reckoning."

Paul E. Hemke and L. E. Williams have been appointed to instructorships in mathematics at the Georgia School of Technology, Atlanta, Ga. They were both graduate students at the University of Chicago.

Professor E. R. Hedrick read a paper at the June meeting of the Society for the Promotion of Engineering Education held at Princeton University, on the subject "The Calculus without Symbols." This paper is published in full by the Society in its volume of proceedings. Special interest attaches to such a discussion in view of the current tendency to introduce the elements of the calculus into secondary instruction.

The first part of the *Encyclopédie des Sciences Mathématiques* appeared in 1904. It was then expected that the entire work could be issued in about 50 parts, each containing about 150 pages. Although more than 30 parts have now been published, it appears likely that more than 50 additional parts will be needed to complete the work. Hence we seem to be farther from the end than they thought they were at the beginning of the publication. More than 20 parts are now in press. It should, however, not be assumed that all of these will appear soon, as some parts have been in press for several years.

Those particularly concerned with the teaching of secondary algebra will be interested in the experiment by Professor E. L. Thorndike of Teachers College, Columbia University, reported in *The Mathematics Teacher* for March. A list of problems in algebra was graded as to difficulty by two hundred teachers of mathematics. There is a satisfactory agreement in the results, notwithstanding some striking differences in estimating the relative difficulty of certain problems. Each of the following was rated all the way from the easiest to the hardest in point of difficulty: (1) "If $a = 6$ and $b = 3$, what does $\sqrt{a}\sqrt{2b}$ equal?" (2) "Find the average midnight temperature for the week in which the daily midnight temperatures were 15, 3, 0, -7, -9, 6, and 17 degrees." (3) "At what time between 6 and 6:30 o'clock are the hands of a watch at right angles to each other?"

The Cambridge University Press publishes "A Shorter Geometry" by Godfrey and Siddons which is written in the spirit of the recommendations of the British Board of Education concerning the teaching of geometry. The course is organized in three stages: (1) Introductory work concerned with the fundamental notions and not primarily designed to give facility in using instruments; (2) Discovery by experiment and intuition of the fundamental facts about angles at a point, parallels, angles of a triangle and polygon, congruent triangles; the accurate use of instruments and the elementary ideas of logical argument; (3) Deductive development of the propositions which can be derived from those propositions discovered by experiment and intuition in the second stage. The proofs of the propositions which were enunciated in the second stage are grouped in an appendix which may be postponed at pleasure or "till they are needed for

examination purposes". It is the expressed belief of the Board that the deductive proofs of the propositions thus postponed are more difficult and less persuasive than those that follow, that difficulties of sequence center here very largely, and that more rapid treatment of these is conducive to better results in subsequent work.

In extension of the summary of recent work on the southerly deviation of a falling body given in the MONTHLY of December, 1913, it may be said that a paper entitled "Deviation of Falling Bodies" by Professor F. R. Moulton of the University of Chicago appears in the June number of the *Annals of Mathematics*. Professor Moulton concludes that when the approximations introduced in the process are properly analyzed, the deviation of a body freely falling a small distance near the earth's surface is found to be equator-ward for all latitudes between $0°$ and $\pm 90°$.

The International Commission on the Teaching of Mathematics met at Paris April 1 to 4 inclusive with an enrolment of 160, half of whom were from France. It was unfortunate that the time of the meeting made impossible a representative attendance from England or America, only six from the former and one from the latter being present. The chief interest centered about the sectional meetings where the subjects of discussion were the reports on (*a*) the results of the introduction of the elementary notions of the calculus into secondary instruction, and (*b*) the mathematical instruction of engineering students. *L'Enseignement* for May 15 gives a very full account of this meeting, including brief summaries of what are said to be admirable reports on the two topics mentioned above. Later notice will be given in these columns of the publication of the complete reports.

The Commission will hold its sessions next year in Munich, at which time it is expected that the reports of the subcommissions of all countries will be complete. It will be remembered that the American reports were all completed for presentation at the time of the meeting of the Commission in Cambridge in 1912, and that, as is probably known to all teachers of mathematics, these may be obtained free on application to the U. S. Bureau of Education, Washington, D. C.

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REMARKS ON KLEIN'S "FAMOUS PROBLEMS OF ELEMENTARY GEOMETRY."

By RAYMOND CLARE ARCHIBALD, Brown University.

No teacher of geometry in a high school should be ignorant of the contents of this little book which appeared in German nearly twenty years ago, but which is also available in an English translation.[1] We here find: (1) the exact statement of the proof of the necessary and sufficient conditions for constructions with ruler and compasses; (2) the proofs of the impossibility of solution, by means of ruler and compasses, of the famous problems: trisection of an angle, duplication of the cube, squaring the circle; (3) full discussion of Gauss's results concerning regular polygons constructible with ruler and compasses; (4) general considerations on algebraic constructions, the integraph, and the geometric construction of π, and other facts of interest in elementary geometry.

While reading this book with a class in geometry for teachers, I had occasion to criticize and elaborate certain parts. In somewhat condensed form, my notes in this connection are given in the following pages.

Gaussian Polygons.—Up to the time of Gauss, no one suspected that it was possible to construct, with ruler and compasses, regular polygons other than those the number of whose sides could be expressed in one of the forms: 2^n, $2^n \cdot 3$, $2^n \cdot 5$, $2^n \cdot 15$. All of these were known to the Greeks. But Gauss proved as early as 1801[2] that whenever a prime number p could be expressed in the form $2^{2^\mu} + 1$, the construction of a regular polygon with p sides was possible by Euclidean methods. It was then apparent that regular polygons not included in the Euclidean series, namely of 17, 257, 65537, . . . sides, could be constructed

[1] F. KLEIN, (*a*) *Vorträge über ausgewählte Fragen der elementar Geometrie*, ausgearbeitet von F. Tägert. Leipzig, Teubner, 1895, 66 pp. +2 Tafeln. (*b*) *Famous Problems of Elementary Geometry* . . . authorized translation of F. Klein's Vorträge . . . , by Wooster Woodruff Beman and David Eugene Smith. Boston, Ginn, 1897, ix+80 pp. (*c*) *Leçons sur certaines questions de géométrie élémentaire* . . . rédaction Française autorisée par l'auteur, par J. Griess. Paris, Vuibert, 1896, 100 pp. There was also an Italian edition published at Turin in 1896, and a Russian edition which appeared in Kasan in 1898.

[2] *Disquisitiones arithmeticæ*, Lipsiæ, 1801, *Werke*, Bd. 1, 2, abdruck 1870, p. 462; French ed. *Recherches Arithmétiques*, Paris, 1807, p. 488; Ger. ed. by Maser, Berlin, 1889, p. 447.

under the same imposed conditions. And indeed Gauss's discussion led to the result,[1] that the *only* regular polygons which it is possible to construct with ruler and compasses, are those the number P of whose sides can be expressed in the form

$$2^{\alpha} \cdot (2^{2^{\alpha_1}} + 1) \cdot (2^{2^{\alpha_2}} + 1) \cdot (2^{2^{\alpha_3}} + 1) \cdots (2^{2^{\alpha_s}} + 1),$$

where $\alpha_1 \cdots \alpha_s$ are distinct positive integers and each $2^{2^{\alpha_i}} + 1$ is a prime. The number of such polygons is small in comparison with the number of regular polygons which can not be constructed with the means employed. As Professor Dickson has pointed out[2] the number of P's up to 100 is 24; up to 300 is 37 (all noted by Gauss); up to 1000 is 52; up to 100000 only 206.

The determination of the number of regular polygons which can be constructed for P less than a given integer is, then, bound up in the determination of the prime numbers p. Now for only 17 values of μ has it been shown whether p is prime or not, namely for the values of μ from 0 to 9 inclusive, and for 11, 12, 18, 23, 36, 38, 73. In the first five of these cases, and in these alone, is p prime. The proof of these five cases was given by Fermat in the seventeenth century. It may well turn out that p is not prime, for $\mu > 4$, although Eisenstein announced[3] the theorem: "There are an infinity of prime numbers of the form $2^{2^{\mu}} + 1$." The result for the case $\mu = 8$ was announced only some five years ago and that for $\mu = 7$ four years earlier. When Klein stated[4] in 1895 that $\mu = 7$ does not give a prime number, he was merely prophetic, while his further statement that for "$\mu = 8$ no one has found out whether we have a prime number or not," must now be modified.

The results already established in this connection may be set forth in tabular form[5] as follows:

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[1] This result was, in effect, stated but not proved by Gauss. *Cf.* the next section.

[2] L. E. DICKSON, "On the Number of Inscriptible Regular Polygons," *Bull. N. Y. Math, Soc.*, Feb., 1894, vol. 3, p. 123.

[3] G. EISENSTEIN, "Aufgaben," *Crelle's Journal*, Vol. 27, 1844, p. 87.

[4] English edition, p. 16; German edition, p. 13; French edition, pp. 26–27.

[5] Largely as by Cunningham and Western in *Proc. London Math. Soc.*, Vol. 1, p. 175, 1903. *Cf. Encyclopédie des sciences mathématiques*, Tome 1, Vol. 3, 1906, p. 51. The authorities for the different results connected with the corresponding case numbers, except those of Fermat, are as follows:

6. L. EULER, *Commentarii Academiae Scientiarum Petrop.*, 1738, Vol. 6 (1732–3), pp. 103–107 p. 104; according to the "Akten" laid before the Academy of St. Petersburg, 26 Sept. 1732. The case $\mu = 5$ was completely factored by Euler about December, 1729. For other editions of this memoir see G. ENESTRÖM, *Verzeichnis der Schriften Leonhard Eulers.* Erste Lieferung, 1910, p. 7.

In his autobiography (Springfield, Mass., 1833, p. 38) the American calculator Zera Colburn records that while on exhibition in London, at the age of 8, he found "by the mere operation of his mind" the factors 641 and 6,700,147 of 4,294,967,297 ($= 2^{32} + 1$). *Cf.* F. D. MITCHELL, "Mathematical Prodigies," *Amer. Journal of Psychology*, Vol. 18, 1907, p. 65.

7. LUCAS, *Comptes Rendus*, Vol. 85, 1878, p. 138; *Amer. Jour. Math.*, Vol. 1, 1878, p. 238; *Recreations mathématiques*, Vol. 2 (2e éd., 1896), pp. 234–5. LANDRY, *Nouv. Corresp. Math.*, Vol. 6, 1880, p. 417.

8. Independent discoverers: WESTERN, *Proc. Lond. Math. Soc.*, (2), Vol. 3, pp. xxi–xxii. Abstract of paper read, 1905; MOREHEAD, *Bull. Amer. Math. Soc.*, Vol. 11, pp. 543–545, abstract, of paper read April 29, 1905.

9. WESTERN AND MOREHEAD, *Bull. Amer. Math. Soc.*, Vol. 16, 1909, pp. 1–6; "Each doing half of the whole work."

Case.	μ	Factors of p.	Discoverer.	Year.
1–5	0–4	All prime	Fermat	1640
6	5	$\left\{\begin{array}{l} 2^7\cdot 5+1=641 \\ 2\cdot{}^7 52347+1=6700417 \end{array}\right\}$	L. Euler	1729
7	6	$\left\{\begin{array}{l} \text{Unknown but composite.} \\ 2^8\cdot 9\cdot 7\cdot 17+1=274177 \\ 2^8\cdot 5\cdot 52562829149+1 \end{array}\right.$	Lucas Landry Landry and Le Lasseur	1878 1880 1880
8	7	Unknown but composite.	A. E. Western, J. C. Morehead	1905
9	8	Unknown but composite	A. E. Western, J. C. Morehead	1909
10	9	$2^{16}\cdot 37+1$	A. E. Western	1903
11	11	$\left\{\begin{array}{l} 2^{13}\cdot 3\cdot 13+1 \\ 2^{13}\cdot 7\cdot 17+1 \end{array}\right\}$	A. Cunningham	1899
12	12	$2^{14}\cdot 7+1$ .... $\left.\begin{array}{l} 2^{16}\cdot 397+1 \\ 2^{16}\cdot 7\cdot 139+1 \end{array}\right\}$	E. Lucas and P. Pervušin A. E. Western	1877 1903
13	18	$2^{20}\cdot 13+1$.........................	A. E. Western	1903
14	23	$2^{25}\cdot 5+1$..........................	P. Pervušin	1878
15	36	$2^{39}\cdot 5+1$..........................	Seelhoff	1886
16	38	$2^{41}\cdot 3+1$..........................	J. Cullen, A. Cunningham, A. E. and F. J. Western	1903
17	73	Unknown but composite.	J. C. Morehead	1906

The labor expended in deriving these results has been enormous. And to the layman who knows nothing of congruences in the theory of numbers, the facts found must seem almost to border on the miraculous. For, even when $\mu = 10$, a case not yet solved, p contains 309 digits; but when $\mu = 36$, p is a number of more than twenty trillion digits. Concerning it Lucas remarked[1] " la bande de papier qui le contiendrait ferait le tour de la Terre." For $\mu = 73$, Ball states that the digits in p " are so numerous that, if the number were printed in full with the type and number of pages used in this book [*Mathematical Recreations*, fifth edition, 1911, 508 pages], many more volumes would be required than are contained in all the public libraries of the world."

In not less than seven places[2] did Fermat refer to $2^{2^\mu} + 1$ as an expression for

10, 12 (Western), 13, 16. *Proc. Lond. Math. Soc.* (2), Vol. 1, 1903, p. 175; abstract of paper read May 14, 1903.

11. A. CUNNINGHAM, *Brit. Assoc. Rept.*, 1899, pp. 653–4.

12, 14. E. LUCAS, *Atti Accad. Torino*, Vol. 13 (1877–8), p. 271 [27 Jan., 1878]. *Mélanges math. astr. acad. Petersb.*, Vol. 5 (1874–81), p. 505, 519 or *Bull. Acad. Pétersb.* (3) vol. 24, 1878, col. 559; (3) Vol. 25, 1879, col. 63; communication by V. Buniakovskij of results, for $\mu = 12$ and 23, found by J. Pervušin, in Nov. 1877 and Feb. 1878.

15. P. SEELHOFF, *Zeitschrift math. u. Phys.*, Vol. 31, 1886, p. 380.

17. J. C. MOREHEAD, *Bull. Amer. Math. Soc.*, Vol. 12, 1906, pp. 449–451.

[1] E. LUCAS, *Théorie des nombres*, Vol. 1, Paris, 1891, p. 51.

[2] Letter dated August [?] 1640 to Frenicle (*Oeuvres de Fermat*, Vol. 2, 1894, p. 206; letter dated 18 October, 1640, to Frenicle (*Oeuvres*, Vol. 2, 1894, p. 208; *Varia Opera*, Toulouse, 1679, p. 162; Brassine's *Précis*, Toulouse, 1853, pp. 142–3); letter dated 25 December, 1640, to Mersenne (*Oeuvres*, Vol. 2, pp. 212–213); " De solutione problematum geometriconum per curvas simplicissimas et unicuique problematum generi proprie convenientes, Dissertatio tripartita " (*Oeuvres de Fermat*, Vol. 1, 1891, pp. 130–131; French translation, Vol. 3, 1896, p. 120; *Varia Opera*, 1679 [reprint, 1861], p. 115); letter dated 29 August, 1654, to Pascal (*Œuvres de Pascal*,

determining a series of prime numbers. The earliest reference is in a letter of August[?], 1640, to Frenicle. He wrote:

"But here is something which pleases me greatly: it is that I am almost persuaded that numbers of the progression 2^{2^0}, 2^{2^1}, 2^{2^2}, 2^{2^3}, $\cdots$, augmented by 1, are prime numbers, as

$$3,\ 5,\ 17,\ 257,\ 65\,537,\ 4\,294\,967\,297,$$

and the following of 20 digits

$$18\,446\,744\,073\,709\,551\,617;\ \text{etc.}$$

I have not an exact demonstration, but I have excluded such a large number of divisors by infallible proofs, and have so many side lights which bear out my thought, that I would find difficulty in convincing myself of error."

In October of the same year Fermat again wrote to Frenicle

"I have not yet demonstrated the exclusion of all divisors in that beautiful proposition which I sent you and which you verified for me with respect to the numbers, 3, 5, 17, 257, 65537, etc For, although I can prove the exclusion of most divisors and show the probability of exclusion for the rest, I am not yet able to demonstrate the necessary truth of the proposition, concerning which however, I have no more doubt at this moment than I had previously. If you have a sure proof you will oblige me by communicating the same to me; for, after that, nothing can keep me back in these matters."

Fourteen years later Fermat had to write to Pascal,

"The demonstration of the proposition is very difficult and I confess to you that I have not yet fully found it; I should not propose that you seek it, had I already reached the goal."

In a somewhat similar vein he demanded a demonstration of Digby in 1658. Indeed, at five different times, last in 1658, Fermat (died 1665) carefully noted that he lacked a rigorous proof that $2^{2^\mu} + 1$ is always a prime number. It is remarkable that he overlooked the fact that this was not prime for $\mu = 5$, since he himself made a remark on the possible factors of numbers of the form $2^m \pm 1$, from which it may be shown that the prime factors of $2^{32} - 1$ must be primes of the form $64n + 1$.[1] From this, Euler's factors 641 and 6700417 could be deduced at once. Fermat's exact statement here, as elsewhere, tends but to confirm the belief in the accuracy of what he wrote with regard to his celebrated theorem (the proof of which has baffled the greatest mathematicians ever since): "I have found for this a truly wonderful proof."[2]

Gauss's Statement of his Polygon Results. In two passages the implication to be drawn from what Klein has written is, that Gauss published a proof that a regular polygon of p sides can not be constructed by ruler and compasses if p is a prime not of the form $2^k + 1$. The passages to which I refer are:[3]

Vol. 4, Paris, 1819, p. 384; *Oeuvres de Fermat*, Vol. 2, 1894, pp. 309–310); letter to Sir Kenelm Digby, sent by Digby to Wallis, 19 June, 1658 (*Oeuvres de Fermat*, Vol. 2, 1894, pp. 402, 404–5; French translation of the Latin, Vol. 3, 1896, p. 314, 316); letter dated August, 1659 to Carcavi, copy sent by Carcavi to Huygens 14 August, 1659 (*Corresp.* de Huygens no. 651; *Oeuvres de Fermat*, Vol. 2, pp. 433–434).

[1] W. W. R. Ball, *Math. Recreations and Essays*, fifth ed., London, 1911, p. 40, is authority for this last statement. I have a distinct impression that it has also been made by some first-class authority, but I have vainly searched Fermat's works for its verification.

[2] *Oeuvres de Fermat*, Vol. 1, 1891, p. 291: "Cujus rei demonstrationem mirabilem sane detexi."

[3] English edition, pp. 2, 16; German edition, pp. 2, 13; French edition, pp. 10, 26.

(1) " Gauss added other cases [to Euclid's] by showing the *possibility* of the division into parts where p is a prime number of the form $p = 2^{2^\mu} + 1$, and the *impossibility* for all other numbers " (the italics here and in (2) are mine); (2) " Gauss extended this series of numbers [Euclid's] by showing that the division is *possible* for every prime number of the form $p = 2^{2^\mu} + 1$ but *impossible* for all other prime numbers and their powers." Now the implication referred to above is not correct, as Professor Pierpont interestingly set forth in his paper " On an undemonstrated theorem of the *Disquisitiones Arithmeticae*."[1] That is, Gauss *did not give a proof* of the " impossibility " referred to in the quotations. But after proving the " possibility " as described above he uses the following words:

"As often as $p-1$ contains other prime factors besides 2, we arrive at higher equations,[2] namely, to one or more cubic equations, if 3 enters once or oftener as a factor of $p-1$, to equations of 5th degree if $p-1$ is divisible by 5, etc. And we can prove with all rigour that these equations cannot be avoided or made to depend upon equations of lower degree; and although the limits of this work do not permit us to give the demonstration here, we still thought it necessary to signal this fact in order that one should not seek to construct other polygons than those given by our theory, as, for example, polygons of 7, 11, 13, 19 sides, and so employ one's time in vain."

To carry Gauss's reasoning further, by supplementing what I have given in the last section, it will be of interest to follow Professor Pierpont. He says:

" Having laid down the theory for polygons of a prime number of sides, Gauss now turns his attention to polygons of any number of sides, $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_\nu^{a_\nu}$ where $p_1, p_2, \cdots$, are the prime factors of n. These he disposes of in a very summary fashion by declaring, without any attempt at proof, that they can be constructed then and only then when

$$\phi(n) = n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_\nu}\right)$$

contains no other factor than 2."

" That this is a sufficient condition follows at once from an easy extension of Gauss's method as developed when n is a prime. It is, however, vastly more important to know that *only* these polygons can be geometrically constructed as thereby the theory of regular polygons, as far as their construction by ruler and compasses is concerned, is made complete. That is, in a given case we can decide whether the polygon is constructible, and in case that it is, Gauss's theory gives us the necessary directions to construct it."

In the first part of his paper Professor Pierpont shows " that the condition which Gauss gave as necessary is in fact such."

Geometrical Constructions of the Regular 17-side. The remark of Klein[3] that we possess as yet no method of construction of the regular polygon of seventeen sides, based upon considerations purely geometrical, is a little curious, since several constructions of this kind have been given. One by Erchinger was indeed reported by Gauss in 1825.[4] The construction is as follows:

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[1] *Bull. Amer. Math. Soc.*, Dec., 1895, Vol. 2, pp. 77–83. *Cf.* also L. E. Dickson in *Monographs on Topics of Modern Mathematics*, London and N. Y., 1911, p. 386.

[2] In his earlier discussion of an inscribed polygon of p sides, Gauss considers the equation $x^p - 1 = 0$ and the resulting equation got by dividing out the factor $x - 1$ where p = prime.

[3] English edition, pp. 24, 32; German edition, pp. 19, 26; French edition, pp. 35, 43.

[4] *Göttingische gelehrte Anzeigen*, Dec. 19, 1825, no. 203, p. 2025; *Werke*, Vol. 2, pp. 186–7. To Art. 365 of the *Disquisitiones Arithmeticae* Gauss added this note in his handwriting: "Circulum in 17 partes divisibilem esse geometrice, deteximus 1796 Mart. 30." [*Cf. Werke*, Vol. 1, p. 476.

Let D, B, G, A, I, F, C, E be points on a line determined by constructions about to be given. Let AB be a line of any length. Produce it both ways to

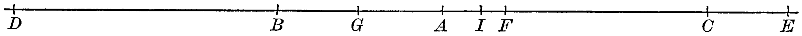

Fig. 1.

C and D so that,

$$AC \times BC = AB \times BD = 4\overline{AB}^2.$$

Further determine the points E, G, on both sides of CA produced so that,

$$AE \times EC = AG \times CG = \overline{AB}^2;$$

and find the point F on the side A of the line BA produced, such that

$$AF \times DF = \overline{AB}^2.$$

Finally divide AE in I so that

$$AI \times EI = AB \times AF,$$

where AI is the smaller, and EI the larger part of AE. Then construct a triangle, in which each of two sides equals AB, the third being equal to AI. About this triangle describe a circle; then AI will be one side of the regular inscribed polygon of seventeen sides.

Gauss particularly remarks that the author gave a purely synthetic proof of this construction.

Another synthetic construction and proof dated "Dublin, 17th October, 1819" was published by Samuel James in the *Transactions of the Irish Academy.*[1] The following earlier solution by John Lowry was Prize Question 410 in *The Mathematival Repository* for 1819:[2] "Draw the radius CO at right angles to the

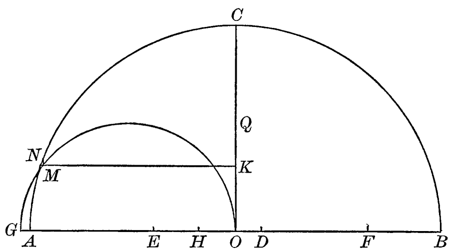

Fig. 2.

diameter AB;[3] on OC and OB, take OQ equal to the half, and OD equal to an eighth part of the radius; make DE and DF each equal to DQ, and EG and FH respectively equal to EQ and FQ; take OK a mean proportional between OH and OQ, and through K draw KM parallel to AB, meeting the semicircle described on OG in M; draw MN parallel to OC cutting the given circle in N, the arc AN is the seventeenth part of the whole circumference."

[1] Vol. 13 (1818), pp. 175–187; paper read Jan. 24, 1820.

[2] New Series, Vol. 4, p. 160. Lowry's proof occupies pp. 160–168.

[3] O is the middle point of AB.

Irrationality of π.—Klein wrote:[1] "After 1770 critical rigour gradually began to resume its rightful place. In this year appeared the work of Lambert: *Vorläufige Kentnisse für die so die, Quadratur . . . des Cirkuls suchen.* Among other matters the irrationality of π is discussed. In 1794 Legendre in his *Éléments de Géométrie* showed conclusively that π and π^2 are irrational numbers." The implication of this note is that Lambert did not discuss the irrationality of π conclusively and that Legendre did. How both of these points of view are essentially incorrect will appear in what follows. Klein was simply reproducing the erroneous statements of Rudio;[2] but after Pringsheim's careful study in 1898,[3] Lambert's proof emerged as "auserordentlich scharfsinnig und im wesentlichen vollkommen einwandfrei," while Legendre's remained "in Bezug auf Strenge hinter Lambert weit zurück."

As in the later proof of the transcendence of π, so here when its irrationality was in question, discussion of e is fundamental. The irrationality of e and e^2 was substantially shown by Euler in 1737[4] and he gave the expression for e as a continued fraction on which Lambert's proofs of the irrationality of e^x, tan x and π rest. Starting with Euler's development[5]

$$\frac{e-1}{2}=\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{10+}\,\frac{1}{14+}\,\frac{1}{18+}\text{ etc.,}$$

Lambert found

$$\frac{e^x-1}{e^x+1}=\frac{1}{2/x+}\,\frac{1}{6/x+}\,\frac{1}{10/x+}\,\frac{1}{14/x+}\text{ etc.,}$$

and since

$$\frac{e^x-1}{e^x+1}=\frac{e^{x/2}-e^{-x/2}}{e^{x/2}+e^{-x/2}}=\tanh\frac{x}{2}=\frac{1}{i}\tan\frac{ix}{2},\quad\text{if}\quad z=\frac{ix}{2},$$

$$\tan z=\frac{1}{1/z-}\,\frac{1}{3/z-}\,\frac{1}{5/z-}\,\frac{1}{7/z-}\,\frac{1}{9/z-}\cdots.$$

He then proved the theorems:

1. *If x is a rational number different from zero, e^x can never be rational.*

For $x = 1$, we have as special case the irrationality of e.

2. *If z is a rational number different from zero, tan z can never be rational.*

For $z = \pi/4$, tan $\pi/4 = 1$, and hence as a special case the irrationality of π.

[1] English edition, p. 59; German edition, p. 46; French edition, p. 72.

[2] F. Rudio: *Archimedes, Huygens, Lambert, Legendre, vier Abhandlungen über die Kreismessung*, Leipzig, 1892, p. 56f. This error is also reproduced by B. Calò in Enriques's *Fragen der Elementargeometrie*, II. Teil, 1907, p. 315; by D. E. Smith in Young's *Monographs on Topics of Modern Mathematics*, 1911, p. 401. The matter was correctly set forth by T. Vahlen in *Konstruktionen und Approximationen*, Leipzig, 1911, pp. 319ff.

[3] A. Pringsheim: "Ueber die ersten Beweise der Irrationalität von e und π." *Sitzungsberichte der mathematisch-physikalischen Classe der k. b. Akademie der Wissenschaften zu München*, Bd. 28, 1899, pp. 325–337.

[4] "De fractionibus continuis," *Comment. acad. de Petrop*, Vol. 9, 1744, p. 108. Presented to St. Petersburg Academy, March, 1737.

[5] L. Euler: *Introductio in analysin infinitorum.* Tomus Primus, Lausannae, 1748, p. 319. This work was finished in 1745; *Cf.* G. Eneström, *Verzeichnis* etc., Erste Lieferung, p. 25.

The part of Lambert's " Vorläufige Kentnisse " to which Klein refers contains some formulae without proof, and no analytical developments, and was rather intended to serve as a popular survey of the treatment of the topic. With it must be considered the scientifically remarkable " Mémoire " of 1767.[1] Here " mit minutiöser Genaugkeit " Lambert proves the convergence of the expression for tan z as a continued fraction. Pringsheim dwells on the " astounding " nature of these considerations at this period in the history of mathematical thought. For of such considerations Legendre was innocent, as well as the great Gauss in his 1812 memoir on hypergeometric series, and others, till a much later period.

" Thus the Lambert memoir contains the *first*, and for many years, the *only* example of what we now consider really rigorous developments of functions as converging continued fractions, in particular, that for tan z given above."

Constructions in General with Ruler and Compasses. While the Greeks built elaborate structures with propositions involving Euclidean methods, Descartes was the first to discuss the question, What geometrical constructions are, and what are not, theoretically possible by these methods? This discussion harks back to the famous, anonymously printed, geometry[2] of 1637, in which the first book was entitled: "The problems which can be constructed by employing circles and straight lines only."[3] Before referring to Klein in this connection, I wish to give some indications of Descartes's argument.[4]

Descartes's Discussion. The book begins with the remark that all the problems of geometry can be reduced to a form in which the only condition of their construction is the determination of the lengths of certain right lines; and that all that is necessary to the determination of the length of a line from sufficient data is the power of interpreting geometrically the five arithmetical operations of addition, subtraction, multiplication, division, and the extraction of roots in such a manner that, the quantities operated on being lengths, the result shall also be a length. This is to be effected by the introduction of the unit of length (if necessary) as a factor or divisor, so as to reduce the construction to finding a fourth proportional to three given lines, or a mean proportional, or several mean proportionals, between two given lines. The actual constructions for multiplication, division, and the extraction of the square root are then given; roots of higher order being reserved for subsequent consideration. Descartes thus disposes of a difficulty which had undoubtedly been felt in the geometric interpretation of expressions above the third degree. But he remarks that any expression which

[1] " Memoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques." Lu en 1767. Printed in 1768 in *Hist. de l'acad. royale des sciences et belles-lettres*, Berlin, Année 1761 (!), pp. 265–322.

[2] Discours *de la methode pour bien conduire sa raison, et chercher la verite dans les sciences. Plus La Dioptrique, Les Meteores et La Geometrie, qui sont des essais de cette Methode.* A Leyde, 1637; in *Oeuvres* de Descartes publiès par Charles Adam et Paul Tannery, Tome VI, Paris, 1902, pp. 1–418.

[3] Adam-Tannery edition, *l. c.*, pp. 369–387.

[4] I follow Professor J. M. Peirce closely in part of his article, "References in Analytic Geometry," *Harvard University Library Bulletin*, Nos. 8, 10, 11, 1878–1879, pp. 157–158, 246–250, 289–290.

denotes a line must be homogeneous and of the first degree, if its value is independent of the choice of the unit. He next speaks of the formation and reduction of equations, in the solution of a problem, and observes that those problems are *plane*, that is, can be solved by straight lines and circles on one flat surface, when the final equation, containing one unknown quantity, is of a degree not higher than the second. The reason of this, of course, is that two circles, or a right line and a circle, intersect in only two points. The actual constructions for the different forms of a quadratic equation are then given. It is important to note that Descartes pays no attention to *negative solutions*. Thus he considers only the three forms,[1] $x^2 = +ax + b^2$, $x^2 = -ax + b^2$, $x^2 = +ax - b^2$, disregarding altogether the form $x^2 = -ax - b^2$, and he constructs only *one* root for each of the first two equations, while he constructs two for the third.[2] Indeed, Descartes seems to have reached a less advanced point on this subject than had already been attained by an earlier writer. Albert Girard, a Dutchman, in his *Invention Nouvelle en l'Algèbre* (Amsterdam, 1629) lays down the true principle of negative quantities, exactly and broadly.

After giving the four forms of construction of a root of a quadratic equation, Descartes makes the important remark that *all the problems of ordinary geometry can be constructed by what is contained in these four figures.* And he continues: "I do not believe that this fact was noted by the ancients; for otherwise they had not taken the trouble to write so many large works where the order alone of their propositions tells us that they did not have the true method to discover them all, but that they have simply collected those which they have met."

As a decisive test of the power of his geometry, Descartes next takes up a problem which Pappus of Alexandria cites at the beginning of the 7th book of his *Mathematical Collections*, stating that neither Euclid nor Apollonius nor any other had been able to give a general solution of it. This problem may be stated, in effect, as follows: "Given in any plane n right lines, and also either n or $(n - 1)$ other right lines, to find the locus of a point such that the product of its distances from the first set of lines measured on lines making any given angles with them, shall be in a given ratio to the product of the distances of the point from the second set of lines measured on lines making given angles with them." With the discussion of this problem, the first book of Descartes's Geometry closes.

Klein's Discussion. In the "Introduction" to Klein's book he announces the fundamental problem stated in the question at the beginning of this section. He

[1] Descartes does not use the sign $=$, but a sign which is perhaps an abbreviation of "aequatur."

[2] In fact, there is no evidence that Descartes perceived that the negative root of an equation (the *false* roots as he calls them) could have any real meaning in a geometric construction; and it is a familiar observation that they generally give a solution of the problem from which they spring, of a kind not contemplated at the outset, and inadmissible if the problem is rigidly interpreted. Later, indeed, in giving constructions for equations of the third and fourth degrees, he exhibits the false roots as well as the true; but in these cases, either of the two sorts of lines may be true roots, according to the sign of one of the coefficients in the equation. Moreover, though Descartes was undoubtedly aware that opposite signs always corresponded to opposite directions, and repeatedly speaks of the possible variations that his constructions thus admit, he does not seem to have apprehended this fact as a *principle*, which might be laid down, once for all, at the outset of the discussion; and he never uses a letter which denotes a quantity as admitting a negative value.

then remarks: "To define sharply the meaning of the word 'construction,' we must designate the instruments which we propose to use in each case. We shall consider," he continues, "(1) straight edge and compasses, (2) compasses alone, (3) straight edge alone, (4) other instruments used in connection with straight edge and compasses."

After two very compact paragraphs, Klein states the following fundamental theorem: "*The necessary and sufficient condition that an analytic expression can be constructed with a straight edge and compasses is that it can be derived from the known quantities by a finite number of rational operations and square roots.*" When this is considered with the following theorem of Chapter I, we have a criterion for identifying the problems under consideration: "*If x, the quantity to be constructed, depends only upon rational expressions and square roots, it is a root of an irreducible equation $\phi(x) = 0$, whose degree is always a power of 2.*" Whence it is shown that if this degree of an irreducible equation is not a power of 2, it cannot be solved by square roots.

Other References. Now the implications in the discussion leading to these theorems are very numerous and no student has mastered all the principles involved until he has approached the subject from other points of view. Foremost to be recommended is Castelnuovo's article adapted from his Projective Geometry[1] for the second part of *Fragen der Elementargeometrie*[2] by Enriques. The articles by Daniele[3] and Giacomini[4] in this same volume should also be carefully studied. Among other results here indicated we find:

That every problem which can be solved with ruler and compasses can also be solved, (*a*) with compasses alone; (*b*) with ruler alone, if we are given a fixed circle with its center in the plane of construction; (*c*) with a ruler alone whose edges are parallel; (*d*) with a ruler alone, whose edges are not parallel, but converge to a point. Many problems can also be solved (*e*) with a ruler and segment-carrier. Theorem (*a*) was first enunciated and proved by Lorenzo Mascheroni (1750–1800) in his famous book *La Geometria del compasso*, Pavia, 1797.[5] An elegant proof by Adler based upon the method of inversion was given in 1890.[6] Mascheroni constructions are treated in English, by Cayley in a paper of 1885,[7] and by Hobson in his recent Presidential Address[8] before the Mathematical Association.

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[1] *Lezioni di geometria analitica e proiettiva*, Roma-Milano, 1905. The article is entitled, "Uber die Lösbarkeit der geometrischen Aufgaben mit den elementaren Instrumenten; Betrachtungen vom Standpunkte der analytischen Geometrie." This and the other articles have also just appeared, in slightly modified form, in *Questioni riguardanti le mathematiche elementari*, raccolte e coordinate da F. Enriques, Volume II, Bologna, 1914. [2] Leipzig, 1907.

[3] "Uber die Lösung der geometrischen Aufgaben mit dem Zirkel."

[4] "Uber die Lösung der geometrischen Aufgaben mit dem Lineal und den linealen Instrumenten: Betrachtungen vom Standpunkte der projektiven Geometrie."

[5] French edition by Carette, Paris, 1798; second edition, 1828. A German translation of the first French edition, "vermehrt mit der Theorie vom Gebrauche des Proportional-Zirkels und mit einer Sammlung zur Uebung von mehr denn 400 rein geom. Sätzen, von J. P. Grüson" was published at Berlin in 1825.

[6] A. Adler, "Zur Theorie der Mascheronischen Konstruktionen," *Situngsberichte der Wiener Akademie*, Bd. 99, Abt. IIa, 1890, p. 910ff.

[7] *Messenger of Mathematics*, vol. 14, 1885, pp. 179–181; *Collected Papers*, Vol. 12, pp. 314–317.

[8] "On geometrical constructions by means of the compass," *Mathematical Gazette*, Vol. 7, March, 1913, pp. 49–54.

To Poncelet (1788–1867) Theorem (*b*) is due,[1] although the result is frequently referred to Steiner who published, in 1833, the little classic, *Die geometrischen Konstruktionen, ausgeführt mittelst der geraden Linie und eines festen Kreises.*[2] But many of the theorems, including the fundamental one, here found, had been already given by Poncelet and Lambert. By omitting to state that the middle point of the given circle must be known[3] Steiner inaccurately formulates the fundamental theorem. In a recent course of lectures Hilbert proved that knowledge of the position of the middle point was essential and suggested as a problem; "How many given circles in a plane are necessary in order to determine with ruler alone, the center of one of them?" In 1912 D. Cauer showed: (1) *If two circles intersect in imaginary points in the finite part of the plane, it is impossible to determine the middle point of either circle with ruler alone;* (2) *A center may be determined if the circles cut in real points, touch, or are concentric.*[3] About the same time J. Grossmann discovered[4] a result which leads us to Theorem

(*f*): Every problem which can be solved with ruler and compasses can also be solved with ruler alone if we are given 3 linearly independent circles, no two concentric, in the plane of construction.

Theorems (*c*) and (*d*) were proved by Adler in 1890.[5] The former is however really implied in Steiner's work referred to above. In the *Foundations of Geometry*[6] by Hilbert, a certain Theorem (*e*) plays a special role and the ideas thus suggested were elaborated by Feldblum in his dissertation.[7] While the theorem and its early applications are due to Hilbert, it is remarkable to find that Lambert introduced the compasses in exactly the same way as the segment-carrier was employed in later times: "Geometrical constructions are all based on the ruler and compasses. . . . The ruler is used simply for drawing straight lines, *and the compasses serve the purpose of marking off lengths on them and acting as carrier, as well as of drawing angles and giving to lines their proper position.*"[8]

To Plato (429–348 B.C.) has been attributed the chief influence in determining that among the instruments which might have been chosen for developing a system of geometry, the ruler and compasses were selected. Already in the tenth century we find the Arabian, Aboûl Wafâ of Bagdad, somewhat limiting, apparently, these means at his disposal by solving problems with ruler as before but with a compass whose arms were open at a constant angle.[9] With such instru-

[1] *Traité des propriétés projectives des figures*, Paris, 1822, pp. 187–190.

[2] An abridgment in French was published by A. Lévy in *Nouv. Annales de Math.*, 1908, pp. 390–409.

[3] Page 68.

[4] *Mathematische Annalen*, Vol. 73, 1912, pp. 90–94. See also Vol. 74, 1913, pp. 462–464.

[5] A. Adler, "Uber die zur Ausführung geometrischer Konstruktionen notwendigen Hilfsmittel," *Sitzungsberichte der Wiener Akademie*, Bd. 99, 1890, Abt. II*a*, p. 846 ff.

[6] Fourth German edition, Leipzig, 1913; English edition by E. J. Townsend, Chicago, 1902.

[7] M. Feldblum, *Ueber Elementar-Geometrische Constructionen.* Diss. Gottingen, Warschau, 1899.

[8] J. H. Lambert, *Beiträge zum Gebrauche der Mathematik und deren Anwendung*, Berlin, 1765, Teil I, pp. 23–24. The latter part of the last sentence quoted has not been translated very literally; the original runs: "und der Zirkel dient, um sie zu fassen und abzutragen, desgleichen auch um Winkel zu ziehen, und den Linien ihre behörige Lage zu geben."

[9] *Woepecke*, "Analyse et extrait d'un recueil de constructions géométriques par Aboûl Wafâ," *Journal Asiatique*, 1855.

ments in the sixteenth century, Cardano, Ferraro, and Tartaglia solved certain problems in Euclid. Without reference to the published solutions of Ferraro and Cardano, Giovanni Battista Benedetti published at Venice in 1553 a treatise entitled: *Resolutio omnium Euclidis problematum, aliorumque ad hoc necessario inventorum, una tantummodo circini data apertura.* For a detailed history of this phase of geometry the reader may consult the monograph by Kutta.[1] In English, I recall but two titles in this connection; the first is a rare pamphlet translated from the Dutch by Joseph Moxon in 1677,[2] the second is an article by the late Dr. J. S. Mackay.[3]

To our series we may now add Theorem

(g): Every problem which can be solved with ruler and compasses, can also be solved with a ruler and one fixed aperture of the compasses.

In concluding this section, it should be remarked with regard to the quotation made above from Klein's Introduction, that his "consideration" of constructions with compasses alone, straight edge alone, and other instruments used in connection with straight edge or compasses, is practically confined to fugitive historical notes on pages 33–34, 47 (English edition).

Concluding Remarks. In the latter part of Chapter IV Klein has made a slight slip (English ed., p. 77; Ger. ed., p. 63; Fr. ed., p. 92). In connection with the equation $y = e^x$, he wrote: " To an algebraic value of x corresponds a transcendental value of y, and conversely." " Conversely " leads us to the statement, to a transcendental value of y corresponds an algebraic value of x. But proof of this has nowhere been given; indeed the result is not true, in general. To correct, delete "conversely" and add: "To an algebraic value of y corresponds a transcendental value of x."

It is to be hoped that the editors of a new edition will be moved

(1) to add a proof that the only regular polygons which it is possible to construct with ruler and compasses are those the number P of whose sides can be expressed in the form

$$2^{\alpha} \cdot (2^{2^{\alpha_1}} + 1) \cdot (2^{2^{\alpha_2}} + 1) \cdot (2^{2^{\alpha_3}} + 1) \cdots (2^{2^{\alpha_s}} + 1)$$

where $\alpha_1 \cdots \alpha_s$ are distinct positive integers and each $2^{2^{\alpha_i}} + 1$ is a prime; α and *one* of the series $\alpha_1, \alpha_2, \cdots \alpha_s$ may be zero.

(2) To at least indicate the connection with continued fractions, and the actual method of determining a and b of the theorem stated on page 17, " if μ and ν are prime to each other, we can always find integers a and b positive or negative, such that $1 = a\mu + b\nu$."

(3) To add the proof that

[1] W. M. Kutta, "Zur Geschichte der Geometrie mit constanter Zirkelöffnung," *Nova Acta Abh. der Kaiserl. Leop.-Carol. Deutschen Akademie der Naturforscher,* Vol. 71, Halle, 1897, pp. 71–101.

[2] *Compendium Euclidis Curiosi: or, geometrical Operations. Shewing how with a single opening of the compasses and a straight ruler all the propositions of Euclid's first five books are performed.* London, 1677. Moxon does not tell us who the author of the original Dutch treatise was.

[3] "Solutions of Euclid's Problems, with a rule and one fixed aperture of the compasses, by the Italian geometers of the sixteenth century," *Proc. Edinb. Math. Soc.,* Vol. 5, 1887, pp. 2–22.

$$(x+1)^{p(p-1)} + (x+1)^{p(p-2)} + \cdots + (x+1)^p + 1 = 0$$

will take the form $$\chi^{p(p-1)} + p \cdot \chi(x),$$

if p is a prime number, and if "$\chi(x)$ is a polynomial with integral coefficients whose constant term is 1" (pages 22–23).

(4) To point out that although the *Conchoid of Nicomedes* is used in the text to trisect an angle, this application of the curve was the discovery of Pappus (about 300 A.D.), and *not* of Nicomedes (about 180 B.C.).[1] Nicomedes used the curve for the duplication of the cube.[2]

(5) To add the indication of proof that a prime number greater than any number we please *exists*. This is needed in the proof of the transcendence of e.

(6) To add, page 61, the proof that $\mathrm{Lim}_{n\doteq\infty}\, x^n/\underline{|n} = 0$.

(7) To make the following corrections:

Page 12, *for* $F(x) = C\cdot[\phi(x)]^\nu$ *read* $F(x) = C_1\cdot[\phi(x)]^\nu$.

Page 14, line 8 should read

$$+ C_1 \sum_{\nu=1}^{\nu=N} \sum_r e^{|k_\nu|} c_r k_\nu^r q_{r,\,k_\nu} + C_2 \sum_{\nu=1}^{\nu=N'} e^{|l_\nu|} c_r' l_\nu^r q_{r l_\nu} + \cdots = 0.$$

Page 34, *for* "with the straight edge and one fixed circle we can solve every quadratic equation," *read* "with the straight edge and one fixed circle, the center being given, we can solve every quadratic equation for which line-segments corresponding to the coefficients are given."

Page 72, line 7, *for* b^{Np} *read* $b^{N'p}$; *for* l_N *read* $l_{N'}$.

Also in the theorem on page 5, some ambiguity would be avoided by setting $\phi(x) = 0$ for $f(x) = 0$.

Note. Since this article was written I have seen a new portion, Bd. III, Heft 5, of the *Encyklopädie der mathematischen Wissenschaften*, published at Leipzig on June 8, 1914. It contains a section by J. Sommer on "Elementare Geometrie vom Standpunkte der neueren Analysis aus," pages 773–858, and about half of the section, pages 859–962, by M. Zacharias on "Elementargeometrie und elementare nicht-Euklidische Geometrie in synthetischer Behandlung." Many parts of this Heft will be of interest in connection with questions discussed above. As to the Gaussian polygons, reference might have been given to paragraph 22 of L. O. Hölder's section of the *Encyklopädie*, published in 1899 and entitled, "Galois'sche Theorie mit Anwendungen."

ON THE TRISECTION OF AN ANGLE AND THE CONSTRUCTION OF REGULAR POLYGONS OF 7 AND 9 SIDES.

By L. E. DICKSON, University of Chicago.

1. Purpose and Plan of this Note. Frequently a wide-awake student who has learned how to bisect any angle asks if every angle can be trisected and, if not, why not; after learning how to construct regular polygons of 3, 4, 5, 6, 8

[1] Pappus, ed. Hultsch, p. 246.

[2] *Cf.* CANTOR, *Vorlesungen über Geschichte der Math.*, Bd. I, 3 Aufl., 1907, p. 351.

and 10 sides, he is apt to ask about the missing ones of 7 and 9 sides. Having several times received a first aid call from teachers of these inquisitive students, the writer would find it convenient to be able to refer to an exposition of these questions which is as elementary as possible. It is the purpose of this note to present such a treatment.

Moreover, it seems necessary that these questions be discussed publicly at regular intervals in order to keep down the number of angle-trisectors, who are partly unable and largely unwilling to understand the standard proofs of the impossibility of these constructions by means of ruler and compasses, but prefer to attempt to make the issue depend upon their own alleged construction involving always a confusing mass of lines and circles and always a child-like error.

With either class of readers, the use of imaginary numbers is not convincing. Hence they are not employed in this note, even though the imaginary roots of unity enter naturally into the questions concerning regular polygons. Moreover, the entire discussion is not beyond a college freshman.

2. The Cubic Equations. In the problem of the duplication of a cube, we take as the unit of length a side of the given cube, and seek the length x of a side of another cube whose volume is double that of the given cube; thus

$$x^3 = 2. \tag{1}$$

In the problem of the trisection of a given angle A, we are given a line of length $\cos A$ and seek a line of length $\cos(A/3)$. For, if we lay off the unit of length AB on one arm of angle A and draw the perpendicular BC to the other arm, the number of units of length in AC is $\cos A$ or $-\cos A$, according as A is an acute or obtuse angle. We employ the well-known trigonometric identity

$$\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3}.$$

Multiply each member by 2 and set $x = 2\cos(A/3)$. Thus

$$x^3 - 3x = 2\cos A.$$

We are to prove that an arbitrary[1] angle A cannot be trisected by ruler and compasses. It suffices to prove this for the angle $A = 120°$. Then $\cos A = -1/2$ and the cubic is

$$x^3 - 3x + 1 = 0. \tag{2}$$

After we have proved that angle 120° cannot be so trisected and hence that angle 40° cannot be constructed by ruler and compasses, it will follow that a regular polygon of nine sides cannot be so constructed, since the angle at the center subtended by one side is $\frac{1}{9}\,360° = 40°$.

Finally, the problem of the construction of a regular polygon of seven sides by ruler and compasses is equivalent to the construction of angle B containing

[1] Certain angles, like $A = 180°$, can be trisected. Since $\cos 60° = 1/2$, the cubic then has the root $x = 1$. Hence this case does not invalidate our general theorem in § 4.

$\frac{360}{7}$ degrees and hence to the construction of a line of length $x = 2 \cos B$. We have

$$\cos 3B = \cos (360° - 3B) = \cos (7B - 3B) = \cos 4B,$$

$$\cos 3B = 4 \cos^3 B - 3 \cos B,$$

$$\cos 4B = 2 \cos^2 2B - 1 = 2 (2 \cos^2 B - 1)^2 - 1.$$

After multiplication by 2 and setting $x = 2 \cos B$, we get

$$x^3 - 3x = 4(\tfrac{1}{2}x^2 - 1)^2 - 2,$$

$$0 = x^4 - 4x^2 + 2 - x^3 + 3x = (x - 2)(x^3 + x^2 - 2x - 1).$$

But $x = 2$ would give $\cos B = 1$, whereas B is acute. Hence[1]

$$x^3 + x^2 - 2x - 1 = 0 \tag{3}$$

3. Our Cubic Equations Have No Rational Roots. Suppose for example that equation (2) has the root a/b, where a and b are integers with no common (integral) divisor greater than unity. Then

$$\frac{a^3}{b^3} - 3\frac{a}{b} + 1 = 0, \qquad \frac{a^3}{b} = 3ab - b^2 = \text{integer}.$$

Thus if $b \neq \pm 1$, b has a divisor greater than unity in common with a, contrary to hypothesis. Hence $b = \pm 1$ and the root is an integer.

If a root x of (2) is an integer, it divides x^3 and $3x$ and hence also the constant term 1, so that $x = \pm 1$. By trial, neither $+ 1$ nor $- 1$ is a root. Hence (2) has no rational root.

The same discussion applies step by step to equation (3). In the case of (1), we must try also the divisors ± 2.

Hence each of our problems has led us to a cubic equation with rational coefficients having no rational root. Each problem is therefore impossible in view of the next theorem.

4. General Theorem. *It is not possible to construct by ruler and compasses a line whose length is a root or the negative of a root of a cubic equation with rational coefficients having no rational root.*

We begin by investigating the nature of a positive number p such that a line of length p can be constructed by ruler and compasses. The ends of this line as well as other points found in the course of the construction are located as the intersections of straight lines and circles. Consider the equations of these lines and circles referred to a fixed pair of rectangular axes, the y-axis not being parallel to any of our straight lines. The equation of any one of our lines is

$$y = mx + b. \tag{1}$$

[1] We can derive this equation and the corresponding ones for other regular polygons without the use of trigonometry, making use only of a theorem on chords in a circle. *Cf.* Dickson, *Annals of Mathematics,* 1894, p. 73.

Another line intersecting this has an equation

$$y = m'x + b'$$

and the coördinates of their point of intersection,

$$x = \frac{b' - b}{m - m'}, \; y = \frac{mb' - m'b}{m - m'},$$

are rational functions of the coefficients of the lines.

To find the coördinates of the intersections of (1) with the circle

$$(x - e)^2 + (y - f)^2 = r^2,$$

we eliminate y and obtain a quadratic equation for x. Thus x, and hence also y, involves no irrationality (besides irrationalities already appearing in m, b, e, f, r) other than a square root.

Finally, the intersections of two circles are given by the intersections of one of them with their common chord, so that this case reduces to the preceding.

Hence the coördinates of the various points located by the construction, and therefore also the length p of the segment joining two of them, are found by a finite number of rational operations and extractions of real square roots, performed upon rational numbers or numbers obtained by such operations. By way of example, note that the side of a regular pentagon inscribed in a circle of radius unity is

$$\tfrac{1}{2}\sqrt{10 - 2\sqrt{5}}.$$

This point settled, consider a cubic equation with rational coefficients and having a constructible root r. Either r is rational or else it involves a real square root. In the latter case, we obtain a second root of the cubic by changing the sign of this square root in the expression for r. Then the third root of the cubic must be rational, since otherwise there would be, as before, a pair of roots in addition to the first pair. Hence in every case the cubic has a rational root, so that the denial of the general theorem stated at the beginning of this section leads to a contradiction. We have merely outlined in a rough way the final step of the proof. The argument in detail is accessible in books by Klein[1] and the writer;[2] it is based upon a systematic classification of the square roots involved in r, but employs only elementary algebraic principles.

The final step in the proof can be made in a few lines by means of the Galois theory of equations, which is based upon the theory of groups.

For a more elaborate elementary discussion of these special problems and the general problem relating to regular polygons, the reader may consult the eighth article in *Monographs on Modern Mathematics*, Longmans, Green and Co., 1911, where further references are given on page 386.

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[1] *Elementarmathematik vom höheren Standpunkte aus*, Leipzig, 1908, vol. 1, p. 125, and 2d ed., 1911.

[2] *Elementary Theory of Equations*, Wiley and Sons, 1914, p. 90.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

The Madison Colloquium Lectures on Mathematics. BY LEONARD EUGENE DICKSON and WILLIAM FOGG OSGOOD. Published by the American Mathematical Society, 501 West 116th St., New York, 1914. 230 pages.

The seventh colloquium of the American Mathematical Society was held in connection with the twentieth summer meeting at the University of Wisconsin during the week of September 8–13, 1913. Professor W. F. Osgood, of Harvard University, gave five lectures on "Topics in the theory of functions of several complex variables," and Professor L. E. Dickson, of the University of Chicago, gave five lectures on "Invariants and the theory of numbers." These lectures have been published by the society as Volume IV of its series of colloquium lectures. The price of the book is $2.00; to members of the society, $1.50. The first three volumes of the series are as follows: I, The Boston Colloquium, 1903. II, The New Haven Colloquium, 1906. III, The Princeton Colloquium, 1909. Before the society became a national organization, a colloquium was held at Evanston, Illinois, in 1893. Professor Klein of Göttingen was the sole speaker. The first edition of this colloquium was exhausted, and a second edition was published by the society under the title: The Evanston Colloquium Lectures on Mathematics. The society held a colloquium at Buffalo in 1896, one at Cambridge in 1898, and one at Ithaca in 1901. The society did not publish the lectures delivered at these colloquia.

W. H. BUSSEY.

Archimedes' Werke. Mit modernen Bezeichnungen herausgegeben und mit einer Einleitung versehen von SIR THOMAS L. HEATH. Deutsch von DR. FRITZ KLIEM. O. Häring, Berlin, 1914.

No greater tribute could be paid to the excellence of the English translation of the works of Archimedes, as made by Heath, than the fact that a German scholar should take this edition as the basis for a retranslation into German. This, too, is the more surprising in view of Heiberg's new and revised Greek and Latin edition of Archimedes' works which is in course of publication, two of the three projected volumes having already appeared. Heiberg's text makes full use of the many additions to our knowledge of Archimedes which have been made in the thirty-odd years since the publication by him of the critical text with Latin translation; seventeen years have elapsed since Heath's work appeared.

Fortunately the German edition has received the careful attention both of Heath and Kliem so that the additions of recent times are incorporated. Thus the newly discovered work of Archimedes, *The Method of Archimedes, concerning theorems (proved by) mechanics—to Eratosthenes,* is incorporated in this volume, based upon the translation published by Heath in 1912 as a supplement to his earlier work. This treatise was found by Heiberg in a manuscript of the tenth century, a palimpsest, now in the Metochion at Constantinople but formerly

from the monastery of the Holy Grave at Jerusalem. This treatise should be familiar to every teacher of elementary mathematics for it gives us clear insight into the methods of work of the greatest mathematician of antiquity. The theorems of the *Method* are entirely within the grasp of any who have studied analytic geometry. The quadrature of a segment of a parabola and the determination of the volume and surface of a sphere are included in the discussion. Besides the translation published by Heath, an English translation is available in the *Monist* for 1909,[1] a German translation in the *Bibliotheca Mathematica,* Vol. 7, third series, pp. 321–363, 1907, and a French description in the *Revue générale des Sciences,* 1907, as well as the more critical Greek edition for philologists in *Hermes,* 1907, Vol. 42, pp. 235–297. How easily an important work may be lost without any trace of it in literature is evident from this treatise which was formerly known only through a chance reference by Suidas to a commentary on it by Theodosius, and the citation by Heron of three of the theorems (and this latter reference first appeared in print in 1903).

Fragments of the *Stomachion* by Archimedes are also preserved in the Constantinople papyrus. Dr. Kliem has added to his valuable work not only these brief passages on this ancient puzzle but also the more complete information preserved in an Arabic translation and published by Suter in Vol. IX of the *Abhandlungen zur Geschichte der Mathematik.* In this geometrical puzzle some fourteen ivory pieces which fit together to make a square are to be arranged in different ways so as to represent a ship, a sword, and other objects. The name of Archimedes has long been associated with this game but only recently has the definite evidence been obtained showing that the great genius of the Greek world did not disdain mathematical puzzles.

This work can be heartily recommended to all students of mathematics, for Dr. Kliem has made available in one volume the works of Archimedes which have come down to us. The book is also a fine specimen of the printer's art. Of corrections necessary I have noted only on page 429, line 14, reference to "Satz 8," instead of "Satz 9."

LOUIS C. KARPINSKI.

Elementary Theory of Equations. By L. E. DICKSON. John Wiley and Sons, New York, 1914. v+184 pages. $1.75.

This book occupies a middle ground in difficulty, being too advanced for the average freshman, but still of an elementary character, suitable for a second course in the theory of equations. It is such a book as may be read with profit by any one who wants an *exact* statement and rigorous proof of the elementary theorems —not involving group-theory or invariants—concerning algebraic equations; a work of value to all teachers of algebra, whether elementary or advanced. In particular every teacher of algebra should read the proof of the fundamental theorem of algebra and the work on graphing; while every teacher of geometry, should read the proofs given in Chap. VIII relating to the trisection of an angle

[1] Republished in pamphlet form by the Open Court Publishing Company.

duplication of the cube, and construction of regular polygons of 7, 9, and 17 sides. An exact treatment of these topics cannot but be of aid to anyone interested in elementary mathematics.

The early introduction of graphing and the use of derivatives in finding " bend points " enable the writer at the beginning of the book to give a discussion of the discriminant of the cubic $x^3 - 3lx + q = 0$, while the work throughout the book is rendered clear by the use of graphs. In the first chapter we are given also the graphical solution of a quadratic. Complex numbers are next introduced in a soul-satisfying way. This chapter should be read by everyone who thinks that complex numbers are " imaginary " and that we gain nothing by their use except to make certain equations have roots.

" The fundamental theorem of algebra " is also treated in a satisfactory way, the graphical proof being clear and elementary. However it tacitly assumes that if a polynomial in x and y is positive at a point A and negative at a second point B, it will vanish at some point between A and B on any curve joining them—a theorem which might have been explicitly stated, particularly since the corresponding theorem for one variable is carefully given (page 13). Also (page 52, line 13) why the word " perhaps "? " A region or perhaps regions " would convey the meaning better. We must have one such region just as assuredly as a region when Y is positive. In fact the statements about these regions are far from clear and are not proved or carefully explained. But when a proof of this important theorem is given which rests on such elementary principles of algebra and graphics as this one does, surely even minor criticism is ill-timed.

The theorem that an integral root of an equation with integral coefficients divides the constant term might well be supplemented by the similar theorem that if an equation with integral coefficients has a fractional root α/β, α must divide the constant term and β the coefficient of the highest power of x. This gives in general a simpler way to find such roots than that given on page 62.

The treatment of symmetric functions is unusually complete and careful.

The reviewer is glad to see in an accessible place a treatment given to the problems of trisecting an angle and duplicating a cube. These puzzle students and often teachers, partly because the problem is not clearly understood, and partly because there is so obviously a solution; and yet their impossibility may readily be made plausible to a student familiar with coördinate geometry and is here rigorously proved in an elementary way. We are also shown why there can be no construction, in the Euclidean sense, of regular polygons of 7 and 9 sides.

The usual theorems for the isolation of the roots are given in Chap. IX, as well as some theorems that are not found in most textbooks. The use of elementary calculus allows a clear treatment and a complete solution of the problem, " given an equation to locate its real roots," while the methods of Chap. X show how to compute them. Besides Horner's well-known method for the numerical computation of roots, Newton's is given and emphasized as one that is effective for non-algebraic as well as for algebraic equations; and Gräffe's little known but very ingenious scheme of solution by forming equations whose roots are powers

of the roots of the given equation, and Lagrange's solution by continued fractions are also explained.

In Chap. XI determinants receive a clear natural treatment, while the subject of resultants and discriminants is carefully and rigorously discussed in the closing chapter.

On the whole in this book there is much to praise and little with which to find fault.

ELIJAH SWIFT.

PROBLEMS AND SOLUTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

SPECIAL NOTICE. In proposing problems and in preparing solutions, contributors will please follow the form established by the MONTHLY, as indicated on the following pages.

In particular, a solution should be preceded by the number of the problem, the name and address of the proposer, the statement of the problem, and the name and address of the solver.

The solution should then be given with careful attention to legibility, accuracy, brevity without obscurity, paragraphing and spacing, having in mind the form in which it will appear on the printed page.

Please use paper of letter size, write on one side only, leaving ample margins, put one solution only on a single sheet and include only such matter as is intended for publication.

Drawings must be made *clearly* and *accurately* and an extra copy furnished on a *separate sheet* ready for the engraver.

Unless these directions are observed by contributors, solutions must be entirely rewritten by the committee or else rejected.

Selections for this department are made two months in advance of publication.

Please send all solutions direct to the chairman of the committee.

MANAGING EDITOR.

ALGEBRA.

When this issue was made up solutions of 410–19 had been received. Solutions of 406 and 420 are desired.

420. Proposed by ELBERT H. CLARKE, Purdue University.

Given the infinite series,

$$\frac{a}{r}+\frac{b}{r^2}+\frac{a+b}{r^3}+\frac{a+2b}{r^4}+\frac{2a+3b}{r^5}+\cdots,$$

in which a and b are any numbers and where each numerator after the first two is the sum of the two preceding numerators. To find the region of convergence and the sum of the series.

This problem is a generalization of one solved in the January number of the MONTHLY.

421. Proposed by C. N. SCHMALL, New York City.

Give a trigonometrical solution of the general quadratic equation.

GEOMETRY.

When this issue was made up solutions of 437–38–39–40–43–45–47–48–54 had been received. Solutions of 427–30–32–33–44–46 are desired.

449. Proposed by H. E. TREFETHEN, Colby College.

Find a tetrahedron with the face angles at one vertex in arithmetical progression and its six edges expressed in integers.

450. Proposed by W. L. WATSON, Moundsville, W. Va.

If three straight lines AA', BB', CC', drawn from the vertices of a triangle ABC to the opposite sides, pass through a common point O within the triangle, then

$$\frac{OA'}{AA'} + \frac{OB'}{BB'} + \frac{OC'}{CC'} = 1.$$

CALCULUS.

When this issue was made up solutions of 354–5–6–7–8–9, 361–2–6–9 had been received. Solutions of 332–39–40–42–64–65 are desired.

370. Proposed by PAUL CAPRON, United States Naval Academy.

The surface of a right circular cone, having the semi-vertical angle α, is cut by two planes which intersect the axis at the same point, one at right angles to the axis, the other making the angle $(90°-\beta)$ with the axis. Show that, if the lateral surface of the right cone is S_1 and that of the oblique cone S_2,

$$S_2 = \sum_1^\infty T_n, \text{ where } T_1 = S_1, \qquad T_{n+1} = T_n \times \frac{2n+1}{2n} (\tan\alpha\tan\beta)^2.$$

371. Proposed by B. F. FINKEL, Drury College.

Prove that the shortest distance between two curves or two surfaces is normal to both.

MECHANICS.

When this issue was made up solutions of 271–4–5, 288–9, 292–3–4–5–7 had been received. Solutions of 268–9, 277–8–9, 286–7 are desired.

298. Proposed by C. N. SCHMALL, New York City.

A person desires to throw a stone so as to strike the greatest possible blow at a point in a smooth vertical wall at a height h above the ground. If his strength is sufficient to throw the stone vertically upwards to a height $2h$, show that he must stand at a distance $2h$ from the wall. (The resistance of the air, and the height of the hand are not taken into account.)

299. Proposed by B. F. FINKEL, Drury College.

A cone rests in two fluids which do not mix, with its vertex downwards and its base in the surface of the upper fluid; to find how much its density must be increased, that it may rest with its base in the common surface of the fluids.

[From Walton's *Hydrostatical Problems.*]

NUMBER THEORY.

When this issue was made up solutions of 207–10–12–13–16–18–20 had been received. Solutions of 189, 191–4–6, 200–5–8–9–11–13–14–15 are desired.

222. Proposed by A. H. HOLMES, Brunswick, Maine.

Find rational values for m and n in order that $(m^2+1)^2/m^2 + (n^2-1)^2/n^2$ may be the square of an integer.

223. Proposed by THOS. E. MASON, Bloomington, Indiana.

Show that

$$\frac{(rst)!}{t!(s!)^t(r!)^{st}}$$

is an integer (r, s, t being positive integers). Generalize to the case of n integers $r, s, t, u, \cdots$.

(Carmichael, *Theory of Numbers*, p. 28.)

SOLUTIONS OF PROBLEMS.

ALGEBRA.

408. Proposed by EMMA M. GIBSON, Drury College, Mo.

Show that if n is a positive integer, the sum of the series

$$1-\frac{2n-1}{\underline{|1}}+\frac{(2n-1)(2n-2)}{\underline{|2}}-\cdots+(-1)^{n-1}\frac{(2n-1)(2n-2)\cdots(n+1)}{\underline{|n-1}}$$

is

$$\frac{(-1)^{n-1}(2n-2)(2n-3)\cdots(n+1)n}{\underline{|n-1}}.$$

Solution by Howard C. Feemster, York College, Neb.

Let

$$S_1=1,\qquad S_2=-\frac{2n-2}{\underline{|1}},\qquad S_3=\frac{(2n-2)(2n-3)}{\underline{|2}},$$

$$S_4=-\frac{(2n-2)(2n-3)(2n-4)}{\underline{|3}}.$$

Assume

$$S_r=\frac{(-1)^{r-1}(2n-2)(2n-3)\cdots(2n-r)}{\underline{|r-1}},$$

then

$$S_{r+1}=\frac{(-1)^{r-1}(2n-2)(2n-3)\cdots(2n-r)}{\underline{|r-1}}$$

$$+\frac{(-1)^{r}(2n-1)(2n-2)(2n-3)\cdots(2n-r)}{\underline{|r}}$$

$$=\frac{(-1)^{r}(2n-2)(2n-3)(2n-4)\cdots(2n-r-1)}{\underline{|r}},$$

which is of the same form as S_r. Hence,

$$S_n=\frac{(-1)^{n-1}(2n-2)(2n-3)(2n-4)\cdots[2n-(n-1)](2n-n)}{\underline{|n-1}}$$

$$=\frac{(-1)^{n-1}(2n-2)(2n-3)(2n-4)\cdots(n+1)n}{\underline{|n-1}},$$

as required.

Note. This is simply half of the expansion, $(1-1)^{2n-1}$, the other half, S_n', is equal to this but opposite in sign. $S_n+S_n'=0$.

Also solved by Horace Olson, A. M. Harding, and Geo. W. Hartwell.

409. Proposed by C. E. GITHENS, Wheeling, W. Va.

Find integral values for the edges of a rectangular parallelopiped so that its diagonal shall be rational.

SOLUTION BY WALTER C. EELLS, United States Naval Academy.

Let x, y, z be the three edges, a the diagonal, and w the hypotenuse of the right triangle of which x and y are sides. Then we seek solutions in integers of the equation $x^2 + y^2 + z^2 = a^2$.

The following method is a generalization of the well known formulas for rational right triangles; namely, if the legs be denoted by $M^2 - N^2$, and $2MN$, and the hypotenuse, therefore, by $M^2 + N^2$, then all integral values of M and N, $(M \neq N)$, will give rational integral sides.

Let $x = 2MN$, $y = M^2 - N^2$, then $w = M^2 + N^2$. This hypotenuse may be thought of under two forms:

$$A. \qquad M^2 + N^2 = m^2 - n^2, \text{ then } z = 2mn, \qquad a = m^2 + n^2;$$

$$B. \qquad M^2 + N^2 = 2m'n', \qquad \text{then } z = m'^2 - n'^2, \; a = m'^2 + n'^2.$$

Under either form it will be necessary to resolve $M^2 + N^2$ into factors, p, q, or $2m'n'$, in all possible ways.

A. $\mathbf{M^2 + N^2 = m^2 - n^2 = (m + n)(m - n) = p \cdot q}$, in which p and q are integers and $p \neq q$. For convenience take $p > q$. Then $m = (p + q)/2$ and $n = (p - q)/2$, whence p and q must be both of form (1) $2k + 1$, or both of form (2) $2k'$.

(1) *p and q both odd.* Then $M^2 + N^2$ must be odd, and M and N one even and one odd. There will be two cases to consider, according as $M^2 + N^2$ is (*a*) prime or (*b*) composite.

(*a*) $M^2 + N^2$ prime. The only factors are itself and unity. For a given N and M there is only one set of values p, q.

Example: $N = 1$, $M = 2$. $x = 2MN = 4$. $y = M^2 - N^2 = 3$. $M^2 + N^2 = 5$. $p = 5$, $q = 1$. $m = (p + q)/2 = 3$. $n = (p - q)/2 = 2$. $z = 2mn = 12$. $a = m^2 + n^2 = 13$. This gives the smallest rational parallelopiped, $(x, y, z, a) \equiv (4, 3, 12, 13)$.

(*b*) $M^2 + N^2$ composite. There are two or more sets of values of p, q.

Example: $N = 1$, $M = 8$, $x = 16$, $y = 63$, $M^2 + N^2 = 65$,

$$\begin{cases} p = 65, \; q = 1; \; m = 33, \; n = 32; \; z = 2112, \; a = 2113; \\ p' = 13, \; q' = 5; \; m' = 9, \; n' = 4; \; z' = 72, \; a' = 97. \end{cases}$$

This gives the two solutions: (16, 63, 2112, 2113), (16, 63, 72, 97).

Remark. It is desirable to so restrict the methods given that only relatively prime values of x, y, z, a, shall appear. Proofs for the statements given to attain this object will not be given, but they are not difficult to supply, based on simple algebra and number theory considerations.

When N and M have a common factor of the form $2k + 1$, $M^2 + N^2$ is composite [(1), (*b*) above], and one of the set of values p, q will have the same common factor $2k + 1$; and $(2k + 1)^2$ will be a common factor of x, y, z, a. This may be divided out giving a reduced solution with x', y', z', a' relatively prime, which will appear elsewhere in the series of relatively prime values found,

e. g., $N = 3$, $M = 6$, gives the solution (36, 27, 108, 117) by the method above, for $p = 15$, $q = 3$. Dividing by 9 there results the solution (4, 3, 12, 13) already given in (1), (*a*).

But *all* combinations where N and M have a common factor $2k + 1$ should

not be excluded,—only those for which in addition p and q have that factor. Thus, for $N = 3$, $M = 6$, as just given, there may be two prime solutions for $(p, q) \equiv (45, 1)$ or $(9, 5)$. They are (36, 27, 1012, 1013) and (36, 27, 26, 53).

Note. In actually computing a, it is easier not to use $a = m^2 + n^2$, but $a = 2mn + q^2$,

$$(a = m^2 + n^2 = 2mn + m^2 + n^2 - 2mn = 2mn + (m - n)^2 = 2mn + q^2),$$

since m and n need not be squared, $2mn$ is already computed, and q is usually small.

(2) *p and q both even.* Then $M^2 + N^2 = 4k$ and M and N are both even, and so have the common factor 2. It is easily shown that x, y, z, a have the common factor 2, and accordingly *no* prime solutions will be given in this case. It can be shown that the reduced forms of all these solutions appear elsewhere in prime form directly,

(a) By method A, (1) if N or M is of form $4k$, and $M^2 + N^2$ prime,

(b) One solution by A, (1) and others by B (below) if N or M is of form $4k$, and $M^2 + N^2$ composite,

(c) By method B if neither N nor M is of form $4k$.

B. $\mathbf{M^2 + N^2 = 2m'n'}$. For convenience the primes are dropped. Then $x = 2MN$, $y = M^2 - N^2$, $z = m^2 - n^2$, $a = m^2 + n^2$ where m and n $(m > n)$ are all possible factors of $(M^2 + N^2)/2$. Since $M^2 + N^2$ must be even, M and N must be (1) both odd or (2) both even.

(1) *N and M even.*

Example: $N = 2$, $M = 4$, $x = 16$, $y = 12$,

$$\begin{cases} m = 10, & n = 1, \; z = 99, \; a = 101, \\ m' = 5, & n' = 2, \; z' = 21, \; a' = 29. \end{cases}$$

Two solutions are (16, 12, 99, 101), (16, 12, 21, 29).

(2) *N and M odd.*

Example: $N = 1$, $M = 3$, $x = 6$, $y = 8$, $z = 24$, $a = 26$.

The example in (2) gives a solution having the common factor 2. It is easy to show that this is always the case when N and M are both odd. To find relatively prime solutions by method B, there should be excluded the following cases. (Details of proof omitted.)

(a) $N = 2k + 1$, $M = 2k' + 1$.

(b) $N = 2k$, $M = 2k'$, for solutions only where m and n are both even. For again it develops that x, y, z, a have the common factor 2,

e. g., $N = 4$, $M = 8$. Exclude for $(m, n) \equiv (20, 2)$ or $(10, 4)$ but retain for $(m, n) \equiv (40, 1)$.

(c) When N and M have a common factor $2k + 1$, for solutions only where m and n have the same common factor $2k + 1$,

e. g., $N = 6$, $M = 12$. Exclude solutions where $(m, n) \equiv (30, 3)$ and $(15, 6)$, but retain them where $(m, n) \equiv (90, 1)$, $(45, 2)$, $(18, 5)$, $(10, 9)$ to secure prime solutions only.

GENERAL REMARKS.

1. The methods of A and B above suffice to determine an indefinite number

of prime solutions of the required equation. Those for which a is less than 100, with the method, A or B, by which they are obtained, are as follows:

x, y, z, a		x, y, z, a	
(4, 3, 12, 13)	A	(12, 5, 84, 85)	A
(16, 12, 21, 29)	B	(64, 48, 39, 89)	B
(24, 32, 9, 41)	B	(16, 63, 72, 97)	A
(36, 27, 28, 53)	A	(56, 33, 72, 97)	A

2. For M and N less than 15, there are twelve groups of two solutions each for the same diagonal a,

$$\text{e. g.,}\ \left\{\begin{matrix} 16, & 63, & 72, & 97 & A \\ 56, & 33, & 72, & 97 & A \end{matrix}\right\} \qquad \left\{\begin{matrix} 96, & 28, & 75, & 125 & B \\ 108, & 45, & 44, & 125 & A \end{matrix}\right\}$$

$$\left\{\begin{matrix} 140, & 171, & 24420, & 24421 & A \\ 220, & 21, & 24420, & 24421 & A \end{matrix}\right\} \qquad \left\{\begin{matrix} 40, & 96, & 153, & 185 & B \\ 72, & 135, & 104, & 185 & A \end{matrix}\right\}$$

3. If x and y are fixed values, in certain cases there are two or more parallelopipeds possible with rational diagonals, according to the number of ways $M^2 + N^2$ may be resolved into factors when composite.

Examples:

(1) $N = 1$, $M = 8$, (16, 63, 2112, 2113),
(16, 63, 72, 97).

(2) $N = 6$, $M = 12$, (144, 108, 19, 181),
(144, 108, 299, 349),
(144, 108, 2021, 2029),
(144, 108, 8099, 8101),

and if composite solutions be allowed the two additional ones

(144, 108, 189, 261),
(144, 108, 891, 909).

4. The method may be further extended to find the diagonal of the four dimensional figure (or n dimensional figure), analogous to the parallelopiped. E. g., to find solutions in integers of $x^2 + y^2 + z^2 + v^2 = a^2$. One such is $(x, y, z, v, a) \equiv (3, 4, 12, 84, 85)$. An easier one is $(a, a, a, a, 2a)$, when a is any integer.

5. For all values of M and N less than 15, there are (unless some error in computation has been made) 125 prime rational parallelopipeds. The smallest is given above. The largest is (420, 29, 88620, 88621).

The following table gives these 125 solutions, arranged according to size of a, and indicating the method by which each was derived. There are 78 solutions found by method A, and 47 by method B.

x	y	z	a		x	y	z	a	
4	3	12	13	A	48	55	2664	2665	A
16	12	21	29	B	364	27	2652	2677	A
24	32	9	41	B	40	96	2703	2705	B
36	27	28	53	A	112	180	2805	2813	B
12	5	84	85	A					
64	48	39	89	B	80	84	3363	3365	B
16	63	72	97	A	36	77	3612	3613	A
56	33	72	97	A	84	13	3612	3613	A
					240	44	3717	3725	B

x	y	z	a	
16	12	99	101	*B*
96	28	75	125	*B*
108	45	44	125	*A*
8	15	144	145	*A*
36	77	132	157	*A*
84	13	132	157	*A*
48	20	165	173	*B*
144	108	19	181	*B*
40	96	153	185	*B*
72	135	104	185	*A*
140	171	60	229	*A*
220	21	60	229	*A*
192	80	105	233	*B*
224	132	69	269	*B*
96	128	231	281	*B*
32	60	285	293	*B*
120	64	273	305	*B*
24	7	312	313	*A*
44	117	300	325	*A*
144	108	299	349	*B*
216	63	272	353	*A*
300	125	228	397	*A*
24	32	399	401	*B*
20	21	420	421	*A*
56	192	375	425	*B*
24	143	408	433	*A*
180	189	380	461	*A*
96	28	621	629	*B*
56	192	609	641	*B*
48	20	675	677	*B*
12	35	684	685	*A*
104	153	672	697	*A*
224	132	651	701	*B*
40	9	840	841	*A*
80	84	837	845	*B*
84	187	828	853	*A*
156	133	828	853	*A*
360	81	800	881	*A*
36	27	1012	1013	*A*
216	63	1000	1025	*A*
32	60	1155	1157	*B*
336	52	1131	1181	*B*
196	147	1188	1213	*A*
48	140	1365	1373	*B*
280	96	1353	1385	*B*
28	45	1404	1405	*A*
264	23	1392	1417	*A*
80	39	3960	3961	*A*

x	y	z	a	
224	132	4221	4229	*B*
120	64	4623	4625	*B*
72	65	4704	4705	*A*
20	99	5100	5101	*A*
48	140	5475	5477	*B*
60	91	5940	5941	*A*
112	15	6384	6385	*A*
96	128	6399	6401	*B*
160	36	6723	6725	*B*
108	45	6844	6845	*A*
336	52	7221	7229	*B*
44	117	7812	7813	*A*
100	75	7812	7813	*A*
144	108	8099	8101	*B*
88	105	9384	9385	*A*
56	192	9999	10001	*B*
24	143	10512	10513	*A*
144	17	10512	10513	*A*
192	80	10815	10817	*B*
140	51	11100	11101	*A*
112	180	11235	11237	*B*
72	135	11704	11705	*A*
132	85	12324	12325	*A*
120	119	14280	14281	*A*
240	44	14883	14885	*B*
52	165	14964	14965	*A*
180	19	16380	16381	*A*
224	132	16899	16901	*B*
104	153	17112	17113	*A*
176	57	17112	17113	*A*
168	95	18624	18625	*A*
28	195	19404	19405	*A*
84	187	21012	21013	*A*
156	133	21012	21013	*A*
280	96	21903	21905	*B*
140	171	24420	24421	*A*
220	21	24420	24421	*A*
216	63	25112	25113	*A*
60	221	26220	26221	*A*
208	105	27144	27145	*A*
336	52	28899	28901	*B*
120	209	29040	29041	*A*
196	147	30012	30013	*A*
180	189	34060	34061	*A*
264	23	35112	35113	*A*
260	69	36180	36181	*A*
252	115	38364	38365	*A*
240	161	41760	41761	*A*

x	y	z	a		x	y	z	a	
64	48	1599	1601	B	312	25	48984	48985	A
160	36	1677	1685	B					
60	11	1860	1861	A	308	75	50244	50245	A
					300	125	52812	52813	A
144	108	2021	2029	B					
16	63	2112	2113	A	364	27	66612	66613	A
56	33	2112	2113	A	360	81	68080	68081	A
96	28	2499	2501	B					
					420	29	88620	88621	A

Also solved in less detail by E. E. Whitford, C. E. Flanagan, and G. I. Hopkins.

GEOMETRY.

Note. The following remark should be made in connection with the solution of Geometry **417** in the September issue:

Three planes determine *either* a point (which may be at infinity if the intersections of the planes in pairs are parallel) *or* a straight line in which the planes are concurrent (which may be at infinity if the planes are parallel). Some of the 220 points required may therefore prove to be replaced by such lines of concurrency of three planes. Whenever the word *line* is used in this solution it refers to a straight line in which three planes are concurrent.

CALCULUS.

337. Proposed by R. P. BAKER, University of Iowa.

Show that for a, b relatively prime integers,

$$\int_0^1 |\cos 2\pi ax + \cos 2\pi bx|\, dx = \frac{2}{\pi ab}\left\{\frac{a+b}{\sin\dfrac{\pi}{a+b}} - \frac{a-b}{\sin\dfrac{\pi}{a-b}}\right\}$$

or

$$= \frac{1}{\pi ab}\left\{(a+b)\cot\frac{\pi}{2(a+b)} - (a-b)\cot\frac{\pi}{2(a-b)}\right\}$$

according as a and b are both odd or one of them is even.

Solution by the Proposer.

Take the first case with $a > b$.

The zeros of the integrand in the path of integration are

$$\xi_{1r} = \frac{2r-1}{2(a+b)}, \qquad r = 1, 3, 5, \cdots (a+b)$$

and

$$\xi_{2s} = \frac{2s-1}{2(a-b)}, \qquad s = 1, 3, 5, \cdots (a-b).$$

If these are $\xi_1, \xi_2, \xi_3, \cdots \xi_{2a}$ in order of magnitude the integral is

$$\frac{1}{\pi ab}\Big[b \sin 2\pi a\xi + a \sin 2\pi b\xi\Big]_{\xi_2,\ \xi_4,\ \cdots\ \xi_{2a}}^{\xi_1,\ \xi_3,\ \cdots\ \xi_{2a-1}}.$$

Consider first the sines of $2\pi b\xi_{1r}$. The rth is preceded by $r-1$ of its own set and by s of the other set where

$$s = \left[\frac{a-b}{2(a+b)}(2r-1)+\tfrac{1}{2}\right];\ [\alpha] \equiv \text{greatest integer in } \alpha.$$

The expression is an integer only when $\xi_{2s} = \frac{1}{2}$ which does not occur in this case.

For the integral we have

$$\sin\pi\left\{\frac{b(2r-1)}{2(a+b)}+(r-1)+\left[\frac{a-b}{2(a+b)}(2r-1)+\tfrac{1}{2}\right]\right\} = \sin\pi(L+[M]) \text{ say.}$$

Now $L+M = 2r-1$ and M is not an integer. Hence $(L+[M]) = 2r-2$ and the sine in question enters the integral positively. The set ξ_{1r} contains all odd multiples of $(a+b)/2$ and b is prime to $(a+b)$ and odd. The angles are then all odd multiples of $\pi/(a+b)$ and all being distinct and in the first two quadrants must be precisely the set

$$\pi k/(a+b), \quad k = 1, 3, 5, \cdots (a+b)-1.$$

The sum of this set of sines contributes to the integral

$$\frac{2}{\pi ab}\left(\frac{a}{\sin\pi/(a+b)}\right).$$

A similar count for the set $\sin 2\pi a\xi_{1r}$, using $L'-[M]$ instead of $L+[M]$, adds a term of the same form, a and b being interchanged.

Similarly the roots of the ξ_{2s} set with a contribute four times the sum of sines of all odd multiples less than the $(a-b)$th of $\pi/(a-b)$. With b the same set occurs with negative sign.

A similar method gives the formula in case 2.

350. Proposed by R. P. BAKER, University of Iowa.

Find a general formula for d^ny/dx^n in terms of d^ky/dt^k and d^kx/dt^k.

SOLUTION BY J. W. CLAWSON Collegeville, Pa.

Let δx, δy be increments of x, y when t takes the increment δt. Then, by Taylor's Theorem,

$$\delta y = \delta x\frac{dy}{dx}+\frac{\overline{\delta x}^2}{\underline{|2}}\frac{d^2y}{dx^2}+\frac{\overline{\delta x}^3}{\underline{|3}}\frac{d^3y}{dx^3}+\cdots, \tag{1}$$

$$\delta y = \delta t\frac{dy}{dt}+\frac{\overline{\delta t}^2}{\underline{|2}}\frac{d^2y}{dt^2}+\frac{\overline{\delta t}^3}{\underline{|3}}\frac{d^3y}{dt^3}+\cdots, \tag{2}$$

$$\delta t = \delta x\frac{dt}{dx}+\frac{\overline{\delta x}^2}{\underline{|2}}\frac{d^2t}{dx^2}+\frac{\overline{\delta x}^3}{\underline{|3}}\frac{d^3t}{dx^3}+\cdots. \tag{3}$$

From (2) and (3) we get

$$\delta y = \left(\delta x\frac{dt}{dx}+\frac{\overline{\delta x}^2}{\underline{|2}}\frac{d^2t}{dx^2}+\cdots\right)\frac{dy}{dt}+\left(\delta x\frac{dt}{dx}+\frac{\overline{\delta x}^2}{\underline{|2}}\frac{d^2t}{dx^2}+\cdots\right)^2\frac{1}{\underline{|2}}\frac{d^2y}{dt^2}+\cdots. \tag{4}$$

Comparing (1) and (4) and equating coefficients of same powers of δx we get

$$\frac{1}{\underline{|n}}\frac{d^n y}{dx^n} = \sum_{k=1}^{k=n} C_k \frac{1}{\underline{|k}}\frac{d^k y}{dt^k},$$

where C_k is the coefficient of $\overline{\delta x}^n$ in the expansion of

$$\left[\delta x\frac{dt}{dx} + \frac{\overline{\delta x}^2}{\underline{|2}}\frac{d^2 t}{dx^2} + \cdots\right]^k.$$

This formula gives $d^n y/dx^n$ in terms of $d^k y/dt^k$ and $d^k t/dx^k$, which is not quite the formula asked for. Compare Greenhill's "D fferential and Integral Calculus," p. 180.

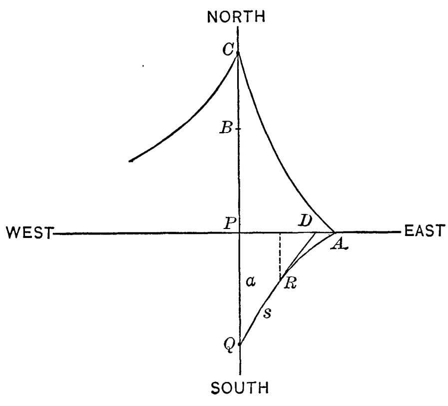
352. Proposed by RICHARD P. LOCHNER, Philadelphia, Pa.

At point P there are n foxes. At Q, a rods south of P, there is a dog. The dog and the foxes are freed at the same instant and run at uniform speeds. Some of the foxes run east, some north, some west and some south. The dog runs first towards the foxes that ran east and always points toward them. He captures one of them and then instantly pursues the pack that ran north. In like manner, when he has captured one of them, he pursues those that ran west, then those that ran south, and then begins over again by pursuing the ones running east. If r is the ratio of the dog's speed to that of a fox, what is the total length of the n curves of pursuit.

(Generalization of a problem published in 1859 in the *Mathematical Monthly.*)

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

Let S be the position of a fox at any time after the chase began and R the corresponding position of the dog. Taking PA for the x-axis and PQ for the y-axis we have



$$QR = r\cdot PD, \quad \text{i. e.,} \quad s = r\left(x - \frac{y}{\frac{dy}{dx}}\right) \quad \text{or} \quad \frac{ds}{dx} = r\left[1 - \frac{\left(\frac{dy}{dx}\right)^2 - y\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^2}\right].$$

Hence,

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = ry\frac{d^2y}{dx^2}\Big/\left(\frac{dy}{dx}\right)^2.$$

Let $\dfrac{dy}{dx} = p$; then $\dfrac{d^2y}{dx^2} = p\dfrac{dp}{dy}$.

Hence, $p\sqrt{1 + p^2} = ry\dfrac{dp}{dy}$ and therefore $\dfrac{dy}{y} = \dfrac{rdp}{p\sqrt{1 + p^2}}$.

Integrating,

$$\log cy = -r \log \frac{[1 + \sqrt{1 + p^2}]}{p}.$$

Hence,

$$cy^{1/r} = \frac{p}{1 + \sqrt{1 + p^2}} \qquad \text{or} \qquad c^2y^{2/r}(1 + p^2) = p^2 - 2cpy^{1/r} + c^2y^{2/r}.$$

Hence,

$$\frac{dy}{dx} = \frac{2cy^{1/r}}{1 - c^2y^{2/r}}. \tag{1}$$

Whence,

$$\frac{1 - c^2y^{2/r}}{y^{1/r}}dy = 2cdx.$$

Integrating,

$$\frac{y^{-1/r+1}}{1 - \frac{1}{r}} - c^2\frac{y^{1/r+1}}{1 + \frac{1}{r}} = 2cx + c'. \tag{2}$$

When

$$x = 0, \qquad y = a, \qquad \frac{dy}{dx} = \infty.$$

Hence, from (1),

$$c = \pm\frac{1}{a^{1/r}},$$

and, from (2),

$$c' = \frac{a^{1-1/r}}{1 - \frac{1}{r}} - \frac{1}{a^{2/r}}\frac{a^{1+1/r}}{1 + \frac{1}{r}}.$$

Hence, from (2),

$$\pm\frac{2x}{a^{1/r}} = \frac{y^{1-1/r}}{1 - \frac{1}{r}} - \frac{1}{a^{2/r}}\frac{y^{1+1/r}}{1 + \frac{1}{r}} - \frac{a^{1-1/r}}{1 - \frac{1}{r}} + \frac{1}{a^{2/r}}\frac{a^{1+1/r}}{1 + \frac{1}{r}}.$$

When

$$y = 0, \qquad x = \pm\frac{a^{1/r}}{2}\left[-\frac{a^{1-1/r}}{1 - \frac{1}{r}} + \frac{1}{a^{2/r}}\frac{a^{1+1/r}}{1 + \frac{1}{r}}\right]$$

$$= \pm\frac{1}{2}\left[\frac{-a}{1 - \frac{1}{r}} + \frac{a}{1 + \frac{1}{r}}\right] = \mp\frac{a}{r\left(1 - \frac{1}{r^2}\right)} = \mp\frac{ar}{r^2 - 1}.$$

Evidently the + sign must be chosen. Hence, $PA = \dfrac{ar}{r^2 - 1}$, and the length of path QA is $\dfrac{ar^2}{r^2 - 1}$.

The north-going pack is now at B, where $PB = PA$. Taking PA as y-axis and PB as x-axis, equations (1) and (2) again hold good. But this time,

when

$$x = 0, \qquad y = \frac{ar}{r^2 - 1} = a_1, \qquad \frac{dy}{dx} = -1.$$

Hence, from (1),

$$-1 = \frac{2ca_1^{1/r}}{1 - c^2a_1^{2/r}}.$$

Whence,

$$c = \frac{1 \pm \sqrt{2}}{a_1^{1/r}}$$

and, from (2),

$$c' = \frac{a_1^{1-1/r}}{1 - \frac{1}{r}} - \frac{3 \pm 2\sqrt{2}}{a_1^{2/r}} \cdot \frac{a_1^{1/r+1}}{1 + \frac{1}{r}}.$$

Hence, from (2),

$$2(1 \pm \sqrt{2})x = \frac{y^{1-1/r}}{1 - \frac{1}{r}} a_1^{1/r} - \frac{3 \pm 2\sqrt{2}}{a_1^{1/r}} \frac{y^{1+1/r}}{1 + \frac{1}{r}} - \frac{a_1}{1 - \frac{1}{r}} + \frac{(3 \pm 2\sqrt{2})a_1}{1 + \frac{1}{r}}.$$

When

$$y = 0, \qquad x = a_1 \frac{(2 \pm 2\sqrt{2}) - (\pm 2\sqrt{2} + 4)\frac{1}{r}}{\left(1 - \frac{1}{r^2}\right) 2(1 \pm \sqrt{2})} = \frac{a_1}{1 - \frac{1}{r^2}} \left\{ 1 \mp \frac{\sqrt{2}}{r} \right\}.$$

Hence,

$$PC = \frac{a_1}{1 - \frac{1}{r^2}} \left\{ 1 + \frac{\sqrt{2}}{r} \right\}.$$

Hence, the length of the path AC is

$$\frac{a_1 r}{1 - \frac{1}{r^2}} \left\{ 1 + \frac{\sqrt{2}}{r} \right\}.$$

The length of each successive curve of pursuit will be found in the same way by multiplying the distance of the dog from P at starting, by

$$\frac{r}{1 - \frac{1}{r^2}} \left\{ 1 + \frac{\sqrt{2}}{r} \right\}.$$

Hence S, the total distance travelled by the dog, is

$$S = r\left[\frac{ar}{r^2 - 1}\left\{1 + \frac{r(r + \sqrt{2})}{r^2 - 1}\left(1 + \frac{r(r + \sqrt{2})}{r^2 - 1}\left(\left(\left(\cdots\right)\right)\right)\right)\right\}\right]$$

$$= \frac{ar^2}{r^2 - 1}\{1 + k + k^2 + \cdots k^{n-1}\}, \qquad \text{where} \qquad k = \frac{r(r + \sqrt{2})}{r^2 - 1}.$$

$$\text{Hence, } S = \frac{ar^2}{r^2 - 1} \frac{1 - k^n}{1 - k} = \frac{ar^2}{(r^2 - 1)^n} \frac{(r^2 - 1)^n - (r^2 + \sqrt{2}r)^n}{r\sqrt{2} + 1}.$$

MISCELLANEOUS QUESTIONS.

EDITED BY R. D. CARMICHAEL.

QUESTIONS.

14. In the process of solving a certain physical problem Professor H. S. Uhler of Yale University was led to the definite integral

$$\int_0^a (a^2 - x^2)xdx \int_{a-x}^{a+x} \frac{e^{-cy}}{y}\, dy,$$

for which he found the value

$$\frac{1}{c^2}\left[a^2 - \frac{3a}{c} + \frac{1}{c}\left(a + \frac{3}{2c}\right)(1 - e^{-2ac})\right],$$

a and c being positive constants. Professor Uhler would like to see how other persons attack the problem of evaluating this integral.

15. We are in receipt of the following communication from Mr. W. E. Heal of Washington, D. C.: "In the Proceedings of the Royal Society of Edinburgh, Vol. VII, p. 144, in some mathematical notes by Professor P. G. Tait, it is stated:

"If $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$.

"This furnishes an easy proof of the impossibility of finding two integers the sum of whose cubes is a cube.

"The writer has failed to see how this 'easy proof' follows and has been unable to find the question discussed or even mentioned in Tait's collected works. Can some reader of the MONTHLY supply the missing link or links?"

REPLY.

10. What use has been made of regular conference periods for assistance to individual students of secondary and college mathematics, and what services may they render?

I. REMARKS BY C. R. MACINNES, Princeton University.

During the last nine years the mathematical department at Princeton has tried with considerable success a scheme for assisting the students in the preparation of their work. Each evening of the week, a class-room or two is kept open and a member of the department is on hand to answer the questions that any student may ask. Attendance is purely voluntary on the part of the students and no attempt is made to keep a check on them. In hard courses like analytics, calculus or mechanics, students use this opportunity for help very freely and, in the main, very fairly. The few who regard it as a chance for free tutoring can soon be attended to, as can the man who obviously has not yet attempted the lesson himself.

An instructor has in this way a chance at the men individually when their heart is in their work, and a good deal can be done. The drawback that we have found comes from the smallness of the department; in a rush it is so much easier to work the student's problem than to point the way out of his immediate difficulty. We have as a usual thing two men on duty each evening and those of us who have done the work regard our evening "in Missouri" as about the most strenuous of the week.

It is the general opinion in our department that the plan has been a considerable help to us.

NOTES AND NEWS.

EDITED BY W. DE W. CAIRNS.

Among recent publications by the University Press of the University of California appears "A complete set of postulates for the logic of classes expressed in terms of the operation 'exception,'" by B. A. BERNSTEIN.

It sometimes happens that a teacher of mathematics desires to examine an article referred to in these columns but does not know where it may be readily obtained. The MONTHLY desires to be as helpful as possible in such matters; an inquiry directed to the chairman of this department, W. D. Cairns, Oberlin, Ohio, will ordinarily bring the desired information.

Dr. E. S. ALLEN, who has been an instructor in mathematics in Dartmouth College, has been appointed to an instructorship in Brown University.

Dr. W. F. SHENTON has been appointed instructor in mathematics at the Johns Hopkins University.

Dr. K. P. WILLIAMS, instructor in mathematics in the University of Indiana, has been advanced to an assistant professorship.

It seems certain that the war in Europe will interfere seriously with the regular receipt in this country of European scientific journals and other publications. Universities and colleges must expect many delays of this sort in the coming months.

For those who have access to the April number of the *Cornhill Magazine* a paper by Professor G. H. Bryan on "Income and Prospects of the Mathematical Specialist" will prove to be very interesting.

The *Mathematical Gazette* for January printed a report entitled "Teaching of calculus in public and secondary schools in the United Kingdom"; this is the report which was presented at Paris in April in response to question A of the International Conference on the Teaching of Mathematics.

The University of Chicago Press has published "Isolation and Measurement of the Electron" by Professor R. A. Millikan of the department of physics of the University of Chicago.

The American Book Company publishes for use this fall a wholly new edition of Raymond's "Plane Surveying for Use in the Class-room and Field." The revised text takes a form more adapted in its size and subject matter to field practice.

Volume 9 of the Supplement of the *Rendiconti del Circolo Matematico di Palermo* contains various addresses given by noted mathematicians during the celebration,

held April 14, 1914, of the thirtieth anniversary of the foundation of the "Circolo Matematico di Palermo." This society has 932 members, being the largest advanced mathematical society in the world, and it publishes a journal which Professor Landau, of Göttingen, Germany, called, in his address on the given occasion, the best mathematical journal in the world.

The April, 1914, number of the *Revista de la Sociedad Matemática Española* opens with an article by Professor G. A. Miller, entitled "Groups, nomenclature and notation."

The article by J. Sommers on elementary geometry from the standpoint of modern analysis, which appeared June 8, 1914, in the *Encyklopädie der Mathematischen Wissenschaften,* is divided into two main parts entitled "Geometry in the narrow sense" and "Spherical Trigonometry." The former is subdivided into two main divisions, entitled "Constructions" and "Applications of the theory of groups." The same number of the *Encyklopädie* contains about one hundred pages of an article by M. Zacharias entitled "Elementary geometry and elementary non-euclidean geometry treated synthetically." The earlier article, entitled "On the development of elementary geometry during the nineteenth century," was prepared by M. Simon for this *Encyklopädie,* but it was not accepted. It appeared as a supplement to the *Jahresbericht der Deutschen Mathematiker-Vereinigung,* 1906.

Some of the numbers of the *Revista de la Sociedad Matemática Española* contain definitions of various mathematical terms arranged in alphabetical order. It may be observed that there is no good modern mathematical dictionary in any language. Such a dictionary in the English language is a desideratum, as it would save much time on the part of the student. The great mathematical encyclopedias which are now being published could be utilized in the preparation of such a dictionary.

Professor W. H. Garrett, for several years the head of the department of mathematics at Baker University, has been granted a year's leave of absence and will spend the time at Harvard University.

Dr. M. O. Tripp, head of the department of mathematics at the Muncie, Indiana, Normal School, has been elected professor of mathematics at Olivet College, Michigan.

At the convocation of the University of Chicago on August 29 thirty-four candidates were given the doctorate of philosophy. Of these, four were in the department of mathematics, namely, Meyer G. Gaba, Forbes B. Wiley, W. V. Lovitt, and Harold R. Kingston. There were also two candidates for the master's degree in mathematics.

In answer to Mr. E. B. ESCOTT's request for integral values of a, b and c which will make

$$2(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)(a^2 + b^2 - c^2)$$

a perfect square, E. Fauquembergue suggests (*L'Intermédiaire* for February) placing the last three factors equal to $2\alpha^2$, $2\beta^2$ and $2\gamma^2$ respectively. The solution is thus reduced to that of the system

$$\beta^2 + \gamma^2 = a^2, \qquad \gamma^2 + \alpha^2 = b^2, \qquad \alpha^2 + \beta^2 = c^2,$$

a problem solved by Euler; the simplest solution is

$$117^2 + 240^2 = 267^2, \qquad 240^2 + 44^2 = 244^2, \qquad 44^2 + 117^2 = 125^2.$$

L'Intermédiaire for January indicates in a brief note the manner in which many proofs in the theory of numbers develop by stages. In 1911 Mr. A. Cunningham reported that he had verified the fact that there is no solution in prime numbers of the congruence

$$2^{n-1} - 1 \equiv 0 \pmod{n^2}$$

for $n < 1{,}000$. He just missed finding a solution of the congruence, for in 1913 W. Meissner proved that there is a solution for $n = 1{,}093$, but that there is no other solution less than 2,000. The February number of the same journal gives the proof for $n = 1{,}093$.

A paper by Professor F. CAJORI entitled "A new marking system and means of measuring mathematical abilities," which was read before the mathematics section of the Central Association of Science and Mathematics Teachers at their last meeting, is printed in full in *Science* for June 12. After maintaining that our high school texts do not differ greatly in the amount of material and difficulty of exercises and that as a consequence an experienced teacher can with due care draw up a "standard" test list of questions, Professor Cajori offers a formula for ordering students according to their ability and a method whereby to rearrange these according to an absolute standard. The formula given is

$$\text{Preliminary Mark} = \frac{M_a + rM_b + sO_a + tO_b + uD}{1 + r + s + t + u},$$

where M represents grades based on memory tests (a) in daily work, (b) in examination, O represents original exercises with the same subdivisions, and D diligence shown, while r, s, t, u are the "weights" given to the last four terms in the numerator, relative to the first; the author suggests $s = 1$, $r = t = u = \frac{1}{3}$, The preliminary marks are then rearranged according to Karl Pearson's theory following the Gaussian law.[1] The paper includes an auxiliary table showing "average differential abilities of pupils chosen at random" from classes of various sizes, as well as a bibliography of the principal material in this branch of statistics.

Science reports that Mr. G. N. WATSON, fellow of Trinity College, Cambridge. has been appointed to a position in pure mathematics in University College, London, to fill the vacancy created by the appointment of Dr. A. N. Whitehead

[1] *Biometrika*, Vol. 1, pp. 390–399.

to the new professorship of applied mathematics in the Imperial College of Science and Technology, London.

Relying upon the list as given in *Science* of August 21, we give the doctorates of philosophy conferred by American universities in mathematics and astronomy during the academic year 1913–14:

Johns Hopkins University—JOSIAH WESLEY GAIN, HARRY CLINTON GOSSARD, BESSIE IRVING MILLER, WALTER FRANCIS SHENTON, MABEL MINERVA YOUNG.

The University of Chicago—WILLIAM CHARLES KRATHWOHL, WILSON LEE MISER, FRANK MARION MORRISON, ELTON JAMES MOULTON, LLOYD ARTHUR HEBER WARREN.

Princeton University—RALPH DENNISON BEETLE, HAIG GALAJIKIAN, JOHN MINOR STETSON.

Harvard University—EDWARD SWITZER ALLEN, RAINARD BENTON ROBBINS.

Cornell University—ROBERT WILBUR BURGESS, ANNA HELEN TAPPAN.

University of Illinois—EDWARD AUGUST THEODORE KIRCHER, LOUIS CLARK MATHEWSON.

University of California—BENJAMIN ABRAM BERNSTEIN, DANIEL WALTER MOREHOUSE.

Clark University—JAMES ATKINS BULLARD.

Columbia University—LYMAN MORSE KELLS.

Indiana University—THOMAS EDWARD MASON.

University of Michigan—SUZAN ROSE BENEDICT.

University of Pennsylvania—STANLEY PULLIMAN SHUGERT.

The titles of the dissertations are given in *Science* for the doctorates in mathematics and all the sciences. Chemistry leads with 71, botany follows with 34, mathematics and zoölogy each have 25, physics has 23. The total in all the sciences is 241, in all other departments combined 261.

The combined departments of mathematics and mathematical astronomy at the University of Chicago conduct club meetings for reports on research bi-weekly during the four quarters of the year. In the summer quarter, meetings are held weekly, the alternate sessions being devoted to the discussion of pedagogical questions. The topics and leaders for the past summer quarter were as follows:

Mr. W. L. HART, "Methods of successive approximation for the solution of equations"; Professor L. E. DICKSON, "The trisection of an angle and other early Greek problems"; Professor D. R. CURTISS, "An example of Gibbs's phenomenon in double Fourier series"; Professors J. W. A. YOUNG and H. E. SLAUGHT, "Books, journals, and associations for teachers of mathematics"; Professor C. N. MOORE, "Some salient features of recent work in the theory of divergent series"; Professor G. A. BLISS, "Mathematical models of the department"; Dr. W. V. LOVITT, "A type of singular points in a transformation of three real variables"; "Open conference on the teaching of mathematics," led by Professor H. E. SLAUGHT, on topics proposed by the students through a question box.

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AN EQUATION BALANCE FOR CLASS-ROOM USE.

By E. W. PONZER, Leland Stanford Junior University.

Of equation balances, many types of which have been described, the one which, in the writer's opinion, possesses the greatest merit for demonstration work in the class-room in connection with the numerical solution of equations is that of Lalanne, presented to the French Academy of Sciences in 1840.[1] The principle on which the balance works is that of moments—a principle fundamental to many machines designed to solve equations.

Perhaps it would be in order here to briefly review the application of the principle so that the mechanical methods by which it is carried out may be clearly understood. Given the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$$

with real coefficients and x_1 as a root of the equation, we may consider each term to be in the form of a moment (positive or negative according to the sign of the coefficient, the root being assumed positive) where a quantity proportional to the coefficient takes the place of the weight and the moment arm is represented by one of the terms $x_1, x_1^2, x_1^3, \cdots x_1^n$. The moment is equal to zero only when x_1 is a root of the equation. A similar condition exists for any other real root of the equation. Imaginary roots would necessarily require a different treatment.

To make use of this principle in a mechanical way all that is necessary is to construct accurately the curves $y = \pm x$, $y = \pm x^2$, $y = \pm x^3$, $\cdots$ and then to use the ordinates corresponding to a real root of the equation as moment arms, together with suitable weights for the coefficients.

The figure on page 284 (showing the standard and right half of the balance) clearly illustrates the construction, which is entirely home-made. The axis on which the root is read is the x-axis. The curves up to $y = \pm x^4$ were plotted between $x = 0$ and $x = 1$ and the brass strips on which the weights are suspended were bent to shape and fastened as shown. The curves all pass through the

[1] *Comptes Rendus*, 1840, Vol. 2, pp. 859–60.

origin and the points (1, 1) or (1, − 1). The guide moves these weights along the curves, keeping them at all times lined up parallel to the axis of the arms. When the proper position is obtained and the balance is in equilibrium the root is read off under the cross-hair.

As will be noticed the balance will not solve equations of a degree above the fourth and the roots must lie between the practical working limits of about 0.4

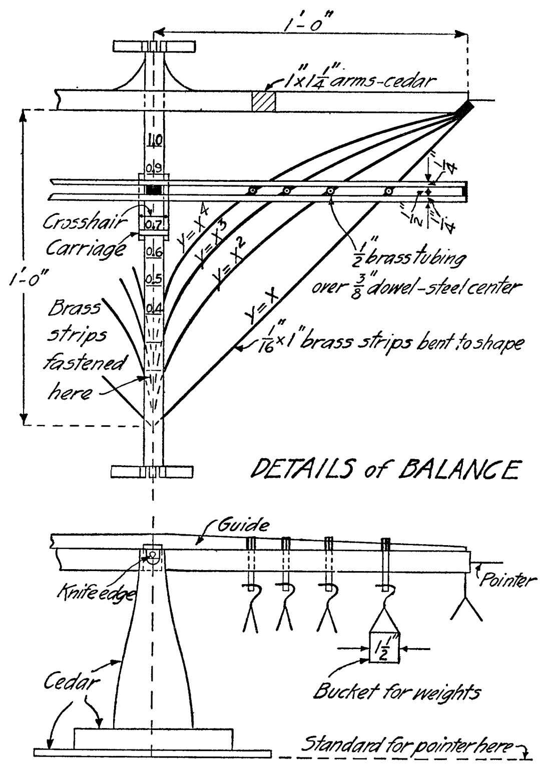

FIG. 1.

to 0.95. However, this limiting of range offers no serious difficulty, for the equation to be solved can always by a suitable transformation $x_1' = kx_1$, $k < 0$ for negative roots, be changed to one whose roots lie within the restricted limits.

For weights 5/16 inch ball bearings (commercial sizes vary by less than 1/10000 of an inch in diameter) serve the purpose well. These are placed in the proper buckets according to the values of the different coefficients. The weight representing a_n is placed in one of the two buckets at the extremities of the balance arms, positive on the right, negative on the left.

The most difficult part of the operation of the balance is to make the stirrups supporting the buckets move freely over the brass curves.

The use of the balance in the solution of an actual problem will illustrate

and help fix the principles according to which it works. Let us take one of the equations used in calibrating the balance. The equation[1]

$$x^3 + 3x^2 - 2x - 5 = 0$$

has a root between 1 and 2. Correct to 3 decimal places it is 1.330. This is outside the range of the balance. Form a new equation whose roots are half those of the original. The transformed equation is

$$8x^3 + 12x^2 - 4x - 5 = 0.$$

The root now sought is 0.665. Place the balance in equilibrium with buckets empty. The buckets on the curves $y = \pm x^4$ are left empty. In the bucket on $y = x^3$ (right) are placed 8 of the ball bearings. Similarly $y = x^2$ (right) is loaded with 12. The line $y = -x$ (left) is loaded with 4, and 5 ball bearings are placed in the bucket at the extremity of the left arm of the balance. Equilibrium is again established by means of the guide which is used to move the stirrups supporting the buckets to their proper position. The cross-hair should then be over 0.665 on the scale, the third figure being estimated. Obviously, the root of the original equation is 1.330.

The balance has stood the test of the class-room for a year and can be depended upon for roots correct to two decimal places, with a close approximation to the third. But more than for its worth in solving equations is it to be valued as an aid in fixing in the students' minds quite a number of the fundamental principles in the theory of equations.

GROUPS OF THE FIGURES OF ELEMENTARY GEOMETRY.

By G. A. MILLER, University of Illinois.

The groups of movements of figures of elementary geometry were considered in an article published in this MONTHLY, volume 10 (1903), page 214. The present note is complementary to this article, but the considerations are based upon entirely different principles. In fact, in the present note we inquire into the substitution group, on the sides or the edges of a figure, which transform this figure into others having the same absolute area or the same absolute volume; while the earlier considerations related to those movements which transform the figure as a whole into itself, but interchange some of its parts.

Let P represent any convex plane polygon of n sides. By interchanging two adjacent sides of P without affecting the other sides we can obtain another convex polygon with the same n sides and with the same absolute area. By a succession of such interchanges it is clearly possible to effect a transposition of any two sides without affecting the absolute area. Since every substitution on the n sides is the product of transpositions, it results that the $n!$ polygons obtained from P in the given manner have the same absolute area, and are conjugate under the symmetric group on their sides.

[1] Fite, *College Algebra*, pp. 192–7.

A necessary and sufficient condition that one of these polygons has a maximal area is that all of them have a maximal area, and hence they can all be inscribed in the same circle, as is also otherwise evident. In fact, this is a direct consequence of the well-known theorem that the circle incloses a larger area than any other plane figure having the same perimeter, since it is possible to construct a convex polygon inscribed in a circle whose sides are equal, in order, to those of any given convex polygon. As a special case there results the elementary theorem that we may interchange, according to any substitution of the symmetric group of degree n, the n sides of any convex polygon inscribed in a circle without affecting the area of this polygon.

The tetrahedron having no two equal edges and no edge which is equal to the sum of two of its edges presents an interesting elementary example from the present point of view. It is evident that the three edges of the base may be interchanged cyclically provided the other three edges are interchanged in the same order. That is, the absolute volume of the tetrahedron will not be affected by eight substitutions of the form $abc \cdot def$, since any one of the four faces may be used as a base. Moreover, if all the edges of a face are left invariant every interchange of the other three edges will in general affect the volume. It will be assumed that the tetrahedron in question has this property.

Hence the substitution group, composed of all the substitutions on the six edges which transform this tetrahedron into another having the same absolute volume, cannot involve more than 24 different substitutions since no more than six such substitutions are possible on the edges of a face. It is very easy to see that there are actually 24 such substitutions. In fact, every face can evidently be transformed into the base, and the edges of the base can be transformed in six different ways without affecting the absolute volume.

In addition to the eight substitutions of the given form there are nine of the form $ab \cdot cd$, and six of the form $abcd \cdot ef$, in addition to the identity. Hence it is easy to identify the substitution group obtained in the given manner as the transitive group of order 24 and of degree 6 which is composed of positive substitutions. This group was denoted by Cayley by the symbol $(+\, abcdef)_{24}$, *Quarterly Journal of Mathematics*, volume 25 (1891), page 81. As is well known it is simply isomorphic with the symmetric group of degree 4, and hence its abstract properties may be directly deduced from those of the latter group.

As there are 720 substitutions on the six edges of the given tetrahedron, and as only 24 of these transform this tetrahedron into one having the same absolute volume, it is generally possible to construct 30 tetrahedrons, having the given six edges, such that no two have the same absolute volume. This fact can also be seen by observing that 20 different triangles can be constructed with these six lines as sides. If we take any one of these triangles as a base it is possible to arrange the other three lines in such a way as to obtain six different tetrahedrons. Hence there result 120 tetrahedrons which have equal absolute volumes, in sets of four, since each face can be used as a base. For greater details along this line the reader may consult an article by Karl Schwering, *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, volume 43 (1912), page 409.

ON A SPECIAL CASE OF THE TETRAHEDRAL COMPLEX.

By D. N. LEHMER, University of California

The teacher of line geometry often stands in need of simple examples to illustrate the general theory. The quadratic complex considered in this paper is a special, but interesting, case of the tetrahedral complex, and while the results might be obtained from the general theory the methods here given are worth noting for their simplicity.

For convenience of expression we shall say that a line in space is at right angles to a flat pencil of rays if it is at right angles to that ray of the pencil which it intersects. Using this definition we may state the theorem:

THEOREM. *The system of lines in space at right angles to a given flat pencil forms a quadratic complex of rays; that is, every point in space is the vertex of a quadratic cone the generating lines of which are lines of the system, and every plane in space contains a conic section the tangent lines of which are lines of the system.*

Let A (Fig. 1) be the center of the flat pencil of rays, and let the plane of the pencil be α. Let also a be any ray of A, and let N be the foot of the perpendicular

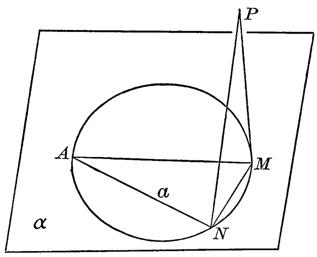

Fig. 1.

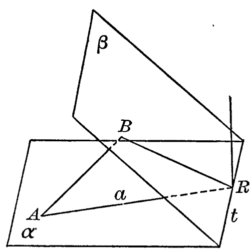

Fig. 2.

from any point P in space upon the line a. Also let M be the foot of the perpendicular from P upon the plane α. Then from elementary geometry ANM is a right angle, and the locus of N is a circle on AM as a diameter. Therefore the rays of the system through P meet the plane α in a circle, and are therefore the generating lines of a quadratic cone.

Further let β (Fig. 2) be any plane in space, and let B be the foot of the perpendicular from A to β. Let t be the line of intersection of α and β, and let R be the point of intersection of any ray a of the pencil A with the line t. Then that line of β which is at right angles to BR and which passes through R is a ray of the system. It follows from the elementary properties of the conic sections that the lines of the system lying in β are tangent to a parabola of which B is the focus, and t the tangent at the vertex. In fact, the lines are seen to be the rays joining corresponding points of two projective point-rows, t and the line at infinity.

Special points of a quadratic complex are such that the cones of rays at these points degenerate into pairs of planes. In the complex described above it is easily seen that the points in α are special. For if P is any point in α then any

line through P in the plane α will be at right angles to some line of A and therefore a line of the system. Also any line through P in a plane at right angles to the line AP will be a line of the system. The cone with vertex P has thus degenerated into a pair of planes. Similarly all the points in the plane at infinity are special. For let P be such a point and a that line of A which is at right angles to PA. Then any line through P in the plane of a and PA belongs to the system. Also any line through P in the plane at infinity belongs to the system. For consider any ray a' of A, and let Q be any point on it. Let a plane at right angles to a' at Q meet PA at P'. Then $P'Q$ is a ray of the system, and if Q moves to infinity P' also moves to infinity in the direction of P. The quadratic cone for any point at infinity is thus seen to degenerate into a pair of planes.

In *Special Planes* of the complex the conic touched by the lines of the system degenerates into a pair of points. This is seen to be the case for any plane through A, and also for any plane through the point at infinity in a direction normal to the plane α. These two points differ from other singular points in that *any* line through them is a line of the system.

Several interesting theorems are easily derived from the above discussion. Thus: *Given two flat pencils in different planes, it is possible to find at most four lines passing through any point of space and at right angles to both pencils.* For the two cones determined by the pencils at any point can have at most four elements in common. An unimportant exception arises when the pencils lie in parallel planes, and the point is chosen on the lines joining their centers. *Given two flat pencils in different planes, it is possible to find at most three finite lines lying in any plane and at right angles to both pencils.* For the parabolas determined by the two complexes in any plane can have at most three tangents in common. Taking account of the infinitely distant elements, however, we may state this and the preceding theorem: *The system of lines at right angles to two flat pencils which lie in different planes is of the fourth order and of the fourth class.*

As a final exercise for the student it is not difficult to prove: *The lines at right angles to two given flat pencils and which meet a given line in space are generators of a ruled quartic surface.*

ON THE USE OF PARTIAL DERIVATIVES IN PLOTTING CURVES FROM THEIR EQUATIONS.

By A. M. KENYON, Purdue University.

The article by M. O. Trip in the January, 1914, number of the MONTHLY on "An Application of Partial Derivatives to the Ellipse," has suggested that some further applications which have proved to be practically effective in tracing curves of the second and third degree may be of interest.

If $f(x, y) = 0$ is the equation (rational and integral in x and y) of a conic section, then $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$ are equations of diameters which bisect all chords parallel to the x- and y-axes respectively, and cut the curve at points where

the tangents are horizontal or vertical. If x be eliminated between the equations $f = 0$ and $\partial f/\partial x = 0$ and y between $f = 0$ and $\partial f/\partial y = 0$, the resulting equations determine the horizontal and vertical tangents respectively.

In a central conic all diameters meet in the center; in a parabola all diameters are parallel to the axis of the curve and any two perpendicular diameters meet on the directrix.

Let $\lambda : \mu$ denote the direction of a straight line which makes with the positive x-axis and with the positive y-axis, the angles whose cosines are respectively $\lambda/\sqrt{\lambda^2 + \mu^2}$ and $\mu/\sqrt{\lambda^2 + \mu^2}$. The direction of the line $ax + by + c = 0$ is $-b : a$ or $b : -a$.

If $f = 0$ is a conic, $\lambda(\partial f/\partial x) + \mu(\partial f/\partial y) = 0$ is a diameter which bisects all chords having the direction $\lambda : \mu$; if the direction $l : m$ of this diameter be determined (in terms of λ and μ), then the equation $\lambda l + \mu m = 0$ determines the directions of the principal axes of the conic; if it is a hyperbola the equation $\mu l - \lambda m = 0$ determines the directions of its asymptotes; if it is a parabola and if $p : q$ is the direction of $\partial f/\partial x = 0$ or $\partial f/\partial y = 0$, then $q(\partial f/\partial x) - p(\partial f/\partial y) = 0$ is the equation of the axis of the curve.

If $f = 0$ is a pair of parallel straight lines, then $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$ represent a line midway between them; if it is a pair of intersecting lines then the equation $\mu l - \lambda m = 0$ determines their directions.

If $f(x, y) = 0$ is the equation (rational and integral in x and y) of a cubic such that no line in the direction $\lambda : \mu$ cuts it in three points, then $\lambda(\partial f/\partial x) + \mu(\partial f/\partial y) = 0$ is a conic (parabola, hyperbola, or pair of straight lines) which bisects all chords of the cubic which have this direction. In case this conic is a hyperbola one of its asymptotes has the direction $\lambda : \mu$ and is an asymptote of the cubic; if the conic is a pair of intersecting lines, one of them has the direction $\lambda : \mu$ and the other bisects all chords of the cubic having this direction; if the conic is a pair of parallel lines, both are asymptotes of the cubic.

EXAMPLES. (1) $2x^2 - 2xy + y^2 - 4x + 2y + 1 = 0$.

$\partial f/\partial x = 2(2x - y - 2)$ and $\partial f/\partial y = -2(x - y - 1)$ cross at the center. Elimination of x between $f = 0$ and $\partial f/\partial x = 0$ gives $y = \sqrt{2}$ and $y = -\sqrt{2}$, horizontal tangents; elimination of y between $f = 0$ and $\partial f/\partial y = 0$ gives $x = 0$ and $x = 2$, vertical tangents. $\partial f/\partial x + \partial f/\partial y = 0$ gives $x = 1$, a diameter bisecting all chords in the direction $1 : 1$; $\partial f/\partial x - \partial f/\partial y = 0$ gives $3x - 2y = 3$, which bisects chords in the direction $1 : -1$.

The direction of $\lambda(\partial f/\partial x) + \mu(\partial f/\partial y) = 0$ is $\lambda - \mu : 2\lambda - \mu$ and the equation $\lambda(\lambda - \mu) + \mu(2\lambda - \mu) = 0$ gives $\lambda : \mu = \sqrt{5} - 1 : 2$ and $\sqrt{5} + 1 : -2$, the directions of the principal axes. By drawing chords across these various diameters from the four points of tangency and other points easily found on the curve, a large number of new points may be located.

(2) $$x^2 - 4xy + y^2 + 6x + 24 = 0.$$

$\partial f/\partial x = 2(x - 2y + 3)$ and $\partial f/\partial y = 2(y - 2x)$ cross at the center. Elimination of x between $f = 0$ and $\partial f/\partial x = 0$ gives $y = -1$, $y = 5$, horizontal tangents;

similarly $x=-2$, $x=4$ are vertical tangents. The equation $\lambda(2\lambda-\mu)+\mu(\lambda-2\mu)=0$ gives $\lambda:\mu = 1:1$ and $1:-1$, the directions of the principal axes; the equation $\mu(2\lambda-\mu)-\lambda(\lambda-2\mu)=0$ gives $\lambda:\mu = 2+\sqrt{3}:1$ and $2-\sqrt{3}:1$, the directions of the asymptotes.

$$x^2+2xy+y^2-2x-6y+1=0. \tag{3}$$

$\partial f/\partial x = 2(x+y-1)$ and $\partial f/\partial y = 2(x+y-3)$ have the same direction $1:-1$. $-\partial f/\partial x - \partial f/\partial y = 0$ gives $x+y=2$, the axis of the curve. Elimination of x between $f=0$ and $\partial f/\partial x = 0$ gives $y=0$, the horizontal tangent. Similarly $x=2$ is the vertical tangent; these tangents cross on the directrix. The focus may now be located.

$$x^3+2y^3-3xy^2-6x-6y=0. \tag{4}$$

$\partial f/\partial x = 3(x^2-y^2-2)$ and $\partial f/\partial y = 6(y^2-xy-1)$. Lines in the two directions $1:1$ and $2:-1$ cut the cubic in two points only. $\partial f/\partial x + \partial f/\partial y = 0$ gives $(x-y+2)(x-y-2)=0$, both asymptotes of the cubic; $2(\partial f/\partial x)-\partial f/\partial y=0$ gives $x^2+xy-2y^2=1$, a hyperbola whose center is at the origin and whose asymptotes are $x-y=0$ and $x+2y=0$, the latter being also an asymptote of the cubic. This hyperbola bisects all chords having the direction $2:-1$.

A METHOD OF SOLVING NUMERICAL EQUATIONS.

By S. A. COREY, Hiteman, Iowa.

The following development of the roots of an equation by Maclaurin's formula applies to both algebraic and transcendental equations, and gives all the roots approximately whenever the conditions involved in the development can be fully complied with.

Let $f(r)=0$ be an equation to be solved, and let a be an approximation to a root r of the equation.

Let us suppose that $f(r)$ is single-valued and analytic in a circle about a as a center and including r. Then

$$f(r) = f(a) + f'(a)(r-a) + \frac{f''(a)}{2!}(r-a)^2 + \frac{f'''(a)}{3!}(r-a)^3 + \cdots.$$

Hence, by the usual formulas for the reversion of series,[1] putting $z = f(r) - f(a)$, we get

$$\begin{aligned} r-a &= \left(\frac{dr}{dz}\right)_0 z + \left(\frac{d^2r}{dz^2}\right)_0 \frac{z^2}{2!} + \left(\frac{d^3r}{dz^3}\right)_0 \frac{z^3}{3!} + \cdots \left(\frac{d^nr}{dz^n}\right)_0 \frac{z^n}{n!} + \cdots \\ &= A_1^{-1}z - A_2A_1^{-3}\frac{z^2}{2!} + (3A_2^2A_1^{-5} - A_3A_1^{-4})\frac{z^3}{3!} + \cdots A_1^{-1}\frac{d}{da}\left(\frac{dr^{n-1}}{dz^{n-1}}\right)_0 \frac{z^n}{n!} + \cdots, \end{aligned} \tag{1}$$

where $A_k = d^kf(a)/da^k$ and $A = f(a)$. In practice it is necessary that the devel-

[1] See, for instance, Goursat, *A Course in Mathematical Analysis*, Vol. 1, §§ 189–190 (first edition).

opment in (1) should be rapidly convergent for the number of terms employed, and the conditions of convergence for the development make it possible to determine the degree of accuracy attained by using a given number of terms.

It will be observed that of all the derivatives A_k in (1), A_1 is the only one which has or can have a negative exponent, and hence the only one which must never have a zero modulus for any value of a as it approaches r. It will also be observed that in order that (1) may be rapidly convergent it is essential that modulus z should be as small as possible, and that modulus A_1 should be as large as possible.

Employing only the terms of (1) preceding the term containing z^4 and putting $z = f(r) - f(a) = -A$, since $f(r) = 0$ for a root r and $A = f(a)$, we get the practical working formula:

$$r = a - AA_1^{-1} - \tfrac{1}{2}A^2A_2A_1^{-3} - \tfrac{1}{2}A^3A_2^2A_1^{-5} + \tfrac{1}{6}A^3A_3A_1^{-4} \cdots, \qquad (2)$$

which is quite well adapted to logarithmic computation for both real and imaginary values of the roots.

Should the degree of accuracy attained by using (2) not be sufficiently great when the first assumed value of a is employed, greater accuracy may be obtained by substituting the new value, and so on, until the required degree of accuracy is attained.

To more clearly indicate the use of the method the following examples will be useful.

Let $f(x) = x^3 + 2x - \sin x - 15 = 0$ be an equation, a real positive root of which, accurate to five decimal places, is to be found.

We know from Sturm's theorem, or can learn by trial, graphic methods or otherwise, that a real root lies between 2 and 3. Therefore assuming that $a = 2$, we get,

$$\begin{aligned} A &= 2^3 + 2\cdot 2 - \sin 2 - 15 &&= -\ \ 3.9093 \\ A_1 &= 3\cdot 2^2 + 2 - \cos 2 &&= +\ 14.4161 \\ A_2 &= 3\cdot 2\cdot 2 + \sin 2 &&= +\ 12.9093 \\ A_3 &= 3\cdot 2 + \cos 2 &&= +\ \ 5.5839 \end{aligned}$$

Substituting in (2), we get $x = 2.244$. Again letting $a = 2.244$, finding new values of A, A_1, A_2, A_3, and substituting in (2), we get $x = 2.243666$, the accuracy of which is determined not by the convergence of (2) but by the values of the sine and cosine as given in the ordinary 6-place tables.

As a second example let us take the quartic equation

$$f(x) = x^4 - 3x^2 + 75x - 10{,}000 = 0,$$

which Merriman gives on page 34 of his *Solution of Equations*, 4th ed., to illustrate Lambert's method. Then taking, as he does, the approximate values of the roots to be the four roots of unity each multiplied by 10, we get

				Modulus	angle
$a=+\ 10$	$-\ 10$	$+10i$	$-10i$	Modulus	angle
$A=+\ 450$	$-1{,}050$	$+\ 300+\ 750i$	$+\ 300-\ 750i$	$\sqrt{652{,}500}$	68° 11′ 55″
$A_1=+4{,}015$	$-3{,}865$	$+\ 75-4{,}060i$	$+\ 75+4{,}060i$	$\sqrt{16{,}489{,}225}$	−(88° 56′ 30″)
$A_2=+1{,}194$	$+\ 1{,}194$	$-1{,}206$	$-1{,}206$	1,206	180° 0′ 0″
$A_3=+\ 240$	$-\ 240$	$+\ 240i$	$-\ 240i$	240	90° 0′ 0″
$A_4=+\ 24$	$+\ 24$	$+\ 24$	$+\ 24$	24	0° 0′ 0″

Substituting in the terms of (2)

$a=$	$+10.00000$	-10.00000	$+10i$	$-10i$
$-AA_1^{-1}=$	$-\ 0.11208$	$-\ 0.27167$	$+0.18330-0.07728i$	$+0.18330+0.07728i$
$-\frac{1}{2}A^2A_2A_1^{-3}=$	$-\ 0.00186$	$+\ 0.01140$	$+0.00428+0.00402i$	$+0.00428-0.00402i$
$-\frac{1}{2}A^3A_2^2A_1^{-5}=$	$-\ 0.00006$	$-\ 0.00095$	$-0.00012-0.00033i$	$-0.00012+0.00033i$
$+\frac{1}{6}A^3A_3A_1^{-4}=$	$+\ 0.00001$	$+\ 0.00021$	$0.00000 \cdot 0.00000$	$0.00000\ \ 0.00000$
$x=\text{sum}=$	$+\ 9.88601$	-10.26101	$+0.18746+9.92641i$	$+0.18746-9.92641i$

Should still greater accuracy be required replace the above values of a by the values of x just found, and so on until the required degree of accuracy is attained.

A FORMULA FOR THE SUM OF A CERTAIN TYPE OF INFINITE POWER SERIES.[1]

By ELBERT H. CLARKE, Purdue University.

Introduction.

The problem to be considered is that of finding a definite, finite formula which will give the sum of any convergent infinite series whose terms are such that their numerators form an arithmetical progression of any order[2] and whose denominators form a geometric progression.

Since the nth term of an arithmetic progression of the kth order may be reduced to the form

$$a_n = b_0n^k + b_1n^{k-1} + \cdots + b_k,$$

our problem is to evaluate the expression

$$T = \sum_{n=1}^{\infty}\frac{b_0n^k + b_1n^{k-1} + \cdots + b_k}{ar^n}$$

in which the b's are independent of n. But this may be written

$$T = \frac{b_0}{a}\sum_{n=1}^{\infty}\frac{n^k}{r^n} + \frac{b_1}{a}\sum_{n=1}^{\infty}\frac{n^{k-1}}{r^n} + \cdots + \frac{b_k}{a}\sum_{n=1}^{\infty}\frac{1}{r^n}$$

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[1] The author wishes to acknowledge criticisms and suggestions from Professors R. D. Carmichael and A. C. Lunn.

[2] See *Text Book of Algebra*, Chrystall, Vol. 1, page 484.

and so the problem is at once reduced to that of finding a formula for

$$\sum_{n=1}^{\infty} \frac{n^k}{r^n} \qquad (|\, r\,| > 1,\ k \text{ a positive integer or zero})$$

which will be denoted in what follows by $S_{k,\,r}$.

The Form of the Sum.

The series $S_{k,\,r}$, being a power series in $1/r$, may be differentiated term by term with respect to r in its region of convergence.

$$\frac{dS_{k,\,r}}{dr} = \sum_{n=1}^{\infty} - \frac{n^{k+1}}{r^{n+1}} = -\frac{1}{r}\sum_{n=1}^{\infty} \frac{n^{k+1}}{r^n}.$$

This gives at once a fundamental relation,

$$S_{k+1,\,r} = -\, r\,\frac{dS_{k,\,r}}{dr}. \tag{1}$$

Furthermore, $S_{0,\,r}$ is none other than the ordinary geometric series, so we have at once

$$S_{0,\,r} = \frac{1}{r-1}$$

and we can derive any particular $S_{k,\,r}$ by k applications of the fundamental relation (1).

By an inspection of the forms of $S_{1,\,r}$; $S_{2,\,r}\,\cdots$; we are led to expect

$$S_{k,\,r} = \frac{r}{(r-1)^{k+1}} F_{k-1}(r),$$

where $F_{k-1}(r)$ is a polynomial of degree $(k-1)$ in r. Applying (1) we obtain, after a few simple reductions,

$$S_{k+1,\,r} = \frac{r}{(r-1)^{k+2}}\left[(kr+1)F_{k-1}(r) + (r-r^2)F'_{k-1}(r)\right]. \tag{2}$$

If $F_{k-1}(r)$ is a polynomial of degree $(k-1)$ in r then the bracketed quantity is a polynomial of degree k in r and may be called $F_k(r)$. In this way the form of $S_{k,\,r}$ is determined except for the particular coefficients occurring in the polynomial $F_{k-1}(r)$.

Determination of the Coefficients.

Let us write

$$F_{k-1}(r) = \alpha_{k-1,\,1}\,r^{k-1} + \alpha_{k-1,\,2}\,r^{k-2} + \alpha_{k-1,\,3}\,r^{k-3} + \cdots + \alpha_{k-1,\,k-1}\,r + \alpha_{k-1,\,k}.$$

Then by carrying out the operations indicated inside the brackets in (2) we obtain

$$\begin{aligned} F_k(r) = \alpha_{k-1,\,1}\,r^k &+ [2\alpha_{k-1,\,2} + k\alpha_{k-1,\,1}]r^{k-1} \\ &+ \cdots + [t\alpha_{k-1,\,t} + (k-t+2)\alpha_{k-1,\,t-1}]r^{k-t+1} \\ &+ \cdots + [k\alpha_{k-1,\,2} + 2\alpha_{k-1,\,k-1}]r + \alpha_{k-1,\,k}. \end{aligned} \tag{3}$$

The following relation is seen to exist between the coefficients of $F_k(r)$ and $F_{k-1}(r)$:

$$\alpha_{k,\,t} = t\alpha_{k-1,\,t} + (k - t + 2)\alpha_{k-1,\,t-1}.$$

By inspection of $F_1(r)$, $F_2(r)$ $\cdots$ we notice that the coefficients are symmetrically arranged and that the first and last coefficients are always 1. By our work we see that the first and last coefficients in $F_{k-1}(r)$ and $F_k(r)$ are the same, therefore they must be 1. Let us assume further a symmetrical arrangement of the coefficients in $F_{k-1}(r)$. That is to say $\alpha_{k-1,\,t} = \alpha_{k-1,\,k-t+1}$. This assumption, in connection with the above recurrence relation between the coefficients of $F_k(r)$ and $F_{k-1}(r)$, gives us immediately

$$\begin{aligned}\alpha_{k,\,t} = t\alpha_{k-1,\,t} + (k - t + 2)\alpha_{k-1,\,t-1} = (k - t + 2)\alpha_{k-1,\,k-t+2} \\ + t\alpha_{k-1,\,k-t+1} = \alpha_{k,\,k-t+2}.\end{aligned} \tag{4}$$

So that symmetry of arrangement of the coefficients in $F_{k-1}(r)$ implies symmetry of arrangement of the coefficients in $F_k(r)$. That the coefficients in $F_k(r)$ are symmetrically arranged follows at once from an inspection of a particular $F_k(r)$, e. g., $F_4(r) = r^4 + 26r^3 + 66r^2 + 26r + 1$.

It is an interesting fact that the sum of the coefficients in $F_k(r)$ is $(k + 1)!$. The reader may easily derive the proof from (3).

As already stated, the *first* coefficient satisfies the relation

$$\alpha_{k,\,1} = \alpha_{k-1,\,1} = \cdots = 1.$$

By carrying out the recurrence relations for the *second* coefficient we obtain

$$\begin{aligned}
\alpha_{2,\,2} &= 2\alpha_{2,\,1} + 2, \qquad (\alpha_{2,\,1} = 1)\\
\alpha_{3,\,2} &= 2^2 \cdot 1 + 2 \cdot 2 + 3,\\
\alpha_{4,\,2} &= 2^3 \cdot 1 + 2^2 \cdot 2 + 2 \cdot 3 + 4,\\
&\cdots\cdots\cdots\\
\alpha_{k,\,2} &= 2^{k-1} \cdot 1 + 2^{k-2} \cdot 2 + 2^{k-3} \cdot 3 + \cdots + 2^2(k-2) + 2(k-1) + k.
\end{aligned}$$

If we write $\alpha_{k,\,2}$ in rows and add by columns we shall obtain without difficulty

$$\alpha_{k,\,2} = 2^{k+1} - (k + 2).$$

To determine the *third* coefficient we may proceed as above making use of the following formulas:

$$\sum_{k=0}^{e} 3^{e-k} \cdot 2^k = 3^{e+1} - 2^{e+1},$$

$$\sum_{k=0}^{e} 3^{e-k} \cdot k^2 = \sum_{k=0}^{e} 3^{e-k} \cdot k + \tfrac{1}{2}\sum_{k=0}^{e} 3^{e-k} - \tfrac{1}{2}(e + 1)^2,$$

$$\sum_{k=0}^{e} 3^{e-k} \cdot k = \tfrac{1}{2}\sum_{k=0}^{e} 3^{e-k} - \tfrac{1}{2}(e + 1).$$

When we have gone through with the necessary work, which is not difficult but is tedious, we obtain the following result

$$\begin{aligned}\alpha_{k,3} &= 2^2[3^{k-2}\cdot 1 + 3^{k-3}\cdot 2\cdot 2 + \cdots + 3(k-2)2^{k-3} + (k-1)2^{k-2}]\\ &\qquad -[3^{k-2}\cdot 1\cdot 3 + 3^{k-3}\cdot 2\cdot 4 + \cdots + 3(k-2)k + (k-1)(k+1)]\\ &= 3^{k+1} - 2^{k+1}(k+2) + \frac{(k+2)(k+1)}{2}.\end{aligned}$$

It is quite possible, of course, to proceed in the same way to obtain the fourth coefficient and so on. But the work will be extremely tedious and will not be likely to render much easier the generalization by inspection which must be made in any case. By an inspection of the results for $\alpha_{k,2}$ and for $\alpha_{k,3}$, together with (3) we are led to assume for the tth coefficient in $F_k(r)$,

$$\alpha_{k,t} = \left[t^{k+1} - (t-1)^{k+1}\binom{k+2}{1} + \cdots + (-1)^{s-1}(t-s+1)^{k+1}\binom{k+2}{s-1} + \cdots + (-1)^{t-1}\binom{k+2}{t-1}\right],$$

which gives the proper forms for $\alpha_{k,2}$ and $\alpha_{k,3}$ above and which further yields proper numerical values for $\alpha_{1,4}, \alpha_{2,4}, \cdots$. All that is needed to complete the induction is a proof that $\alpha_{k,t}$ will satisfy the recurrence relation (4). That is to say,

$$t^{k+1} - (t-1)^{k+1}\binom{k+2}{1} + \cdots + (-1)^{s-1}(t-s+1)^{k+1}\binom{k+2}{s-1} + \cdots + (-1)^{t-2}2^{k+1}\binom{k+2}{t-2} + (-1)^{t-1}\binom{k+2}{t-1}$$

must be equal to

$$\begin{aligned} t&\left[t^k - (t-1)^k\binom{k+1}{1} + \cdots + (-1)^{s-1}(t-s+1)^k\binom{k+1}{s-1} + \cdots \right.\\ &\left. + (-1)^{t-1}\binom{k+1}{t-1}\right] + (k-t+2)\left[(t-1)^k - (t-2)^k\binom{k+1}{1} + \cdots \right.\\ &\left. + (-1)^{s-1}(t-s)^k\binom{k+1}{s-1} + \cdots + (-1)^{t-2}\binom{k+1}{t-2}\right].\end{aligned}$$

Carrying out the work, the latter expression becomes

$$\begin{aligned} t^{k+1} &- (t-1)^k t\binom{k+1}{1} + \cdots + (1)^{s-1}(t-s+1)^k t\binom{k+1}{s-1} + \cdots + (-1)^{t-1}t\binom{k+1}{t-1}\\ &+ (t-1)^k(k-t+2) + \cdots + (-1)^{s-2}(t-s+1)^k(k-t+2)\binom{k+1}{s-2} + \cdots\\ &+ (-1)^{t-2}(k-t+2)\binom{k+1}{t-2}.\end{aligned}$$

Combining like terms, we have

$$t^{k+1}-(t-1)^k[t(k+1)-k+t-2]+\cdots+(-1)^{s-1}(t-s+1)^k\binom{k+1}{s-2}$$
$$\times\left[t\frac{k-s+3}{s-1}-k+t-2\right]+\cdots+(-1)^{t-1}\binom{k+1}{t-2}\left[t\frac{k-t+3}{t-1}-k+t-2\right],$$

which reduces to the left member above:

$$t^{k+1}-(t-1)^{k+1}\binom{k+2}{1}+\cdots+(-1)^{s-1}(t-s+1)^{k+1}\binom{k+2}{s-1}+\cdots$$
$$+(-1)^{t-1}\binom{k+2}{t-1}.$$

If we replace k by $k-1$ we shall have the tth term in $F_{k-1}(r)$. Hence we have the desired formula for $S_{k,r}$:

$$S_{k,r}=\sum_{n=1}^{\infty}\frac{n^k}{r^n}=\frac{r}{(r-1)^{k+1}}\Bigg\{r^{k-1}+[2^k-(k+1)]r^{k-2}+\cdots$$
$$+\left[t^k+\cdots+(-1)^{s-1}(t-s+1)^k\binom{k+1}{s-1}+\cdots\right.$$
$$\left.+(-1)^{t-1}\binom{k+1}{t-1}\right]r^{k-t}+\cdots+1\Bigg\}.$$

The general series of the introductory paragraph may now be written

$$T=\frac{b_0}{a}S_{k,r}+\frac{b_1}{a}S_{k-1,r}+\cdots+\frac{b_r}{a}S_{0,r}.$$

BOOK REVIEWS.

Memorabilia Mathematica. By ROBERT EDOUARD MORITZ. vii+410 pages. The Macmillan Co., New York, 1914. $3.00 net.

In the Memorabilia Mathematica we are presented with a collection of more than 1100 quotations pertaining to many phases of mathematics and to the life and thought of mathematicians. The page preceding the title page is graced by two quotations from Goethe and Emerson. They read: "Alles Gescheite ist schon gedacht worden; man muss nur versuchen es noch einmal zu denken," and "A great man quotes bravely, and will not draw on his own invention when his memory serves him with a word as good."

We may safely infer that in these quotations, if not interpreted too literally, the author expresses a conviction and implies a purpose. The intrinsic value of the ideas expressed in the Memorabilia shows that the author's conviction is well founded. His purpose is realized in bringing this wealth of ideas within easy reach of mathematical and non-mathematical students.

The volume presents a great range of subjects. It deals not only with the nature and value of mathematics, with the relation of mathematics to logic, philosophy, and science, and with mathematics as a fine art and as a language; but it also introduces a strong personal, human element and deals with the mathematician. In a work of this kind we expect to find many mathematical authors quoted; but are hardly prepared to meet, in addition to the mathematical authors, the names of many men famous in other walks of life. But such is the case. We meet Benjamin Franklin and Daniel Webster, John Locke and Francis Bacon, Goethe and Froebel, Napoleon and Voltaire, and many others.

The book is very suggestive and stimulating. Many viewpoints that are presented contain a challenge to the thinking mind. In dealing with the nature of mathematics we thus find, from the pen of Klein, "Mathematics is fundamentally the science of self evident things," to which Pringsheim replied: "For the majority of the cultured, even of scientists, mathematics remains the science of the incomprehensible." The remark of Huxley certainly contains a challenge: "Mathematics is that subject which knows nothing of experiment, nothing of induction, nothing of causation."

In the light of the fact that mathematics is often spoken of as the science that deals largely with logical processes, and that draws necessary conclusions, the following from Voltaire is significant: "There is an astonishing imagination, even in the science of mathematics. . . . We repeat, there was far more imagination in the head of Archimedes than in that of Homer." From another writer, bearing on the same subject and emphasizing other qualities than that of logic, comes the dictum: "The whole of mathematics consists of a series of aids to the imagination in the process of reasoning." Keyser introduces us to a significant, interesting encounter: "When the greatest of American logicians, speaking of the powers that constitute the born geometer, had mentioned conception, imagination, and generalization, he paused. Thereupon from the audience there came the challenge, 'What of reason?' The instant response, not less just than brilliant, was: 'Ratiocination—that is but the smooth pavement on which the chariot rolls.'"

The Memorabilia has rendered a real service in presenting quotations of sufficient length to illuminate and give a setting for ideas, which, without such a setting, are incomprehensible and misleading. Thus the quotation of Russell, at times quoted with startling effect, that "Mathematics may be defined as the subject in which we never know what we are talking about, or whether what we are saying is true," might be taken as that of a cynic or an iconoclast; but the Memorabilia presents this and other passages with enough of the context, so that they come to us as significant, constructive contributions.

The Memorabilia also presents us with many a human touch, and throws a light, not to be found in purely mathematical books, on many well known men. To take but one case, let us consider Newton for a moment. We find

> "Nature and nature's law lay hid in night:
> God said, 'Let Newton be!', and all was light."

and

> "A monument to Newton! A monument to Shakespeare! Look up to heaven—look into the Human Heart. Till all the planets and the passions—the affections and the fixed stars are extinguished—their names cannot die."

In contrast to this, on the other hand, we are told that Newton "could not readily make up a common account: and, when master of the mint, used to get somebody else to make up his accounts for him." Another side light is the mention of his relation to astrology. "There was a time when Newton was possessed with the old fooleries of astrology; and another when he was so far gone in those of chemistry as to be upon the hunt after the philosopher's stone."

Of the modesty of this great man we are told "If I have seen farther than Descartes, it is by standing on the shoulders of giants." And again: "Newton could not admit that there was any difference between him and other men, except in the possession of such habits as . . . perseverance and vigilance. When he was asked how he made his discoveries, he answered, 'By always thinking about them'; and at another time he declared that if he had done anything, it was due to nothing but industry and patient thought: 'I keep the subject of my inquiry constantly before me, and wait till the first dawning opens gradually, by little and little, into a full and clear light.'"

The devotees in many ages have sounded the praises of mathematics. From Euripides the Memorabilia presents us with this contribution: "Mighty are numbers, joined with art resistless"; and from Plato comes a passage that contains a claim for arithmetic which has been heard for mathematics in general in more modern days, for we read: "No single instrument of youthful education has such mighty power, both as regards domestic economy and politics, and in the arts, as the study of arithmetic. Above all, arithmetic stirs up him who is by nature sleepy and dull, and makes him quick to learn, retentive, shrewd, and aided by art divine he makes progress quite beyond his natural powers." Another note of praise is sounded in the words: "If the Greeks had not cultivated Conic Sections, Kepler could not have superseded Ptolemy; if the Greeks had cultivated Dynamics, Kepler might have anticipated Newton."

The relation of mathematics to our every day experiences is expressed in such passages as the following: "Mathematics, the science of the ideal, becomes the means of investigating, understanding and making known the world of the real." And again: "Geometry in every proposition speaks a language which experience never dares to utter; and indeed of which she but half comprehends the meaning. Experience sees that the assertions are true, but she sees not how profound and absolute is their truth. She unhesitatingly assents to the laws which geometry delivers, but she does not pretend to see the origin of their obligation. She is always ready to acknowledge the sway of pure scientific principles as a matter of fact, but she does not dream of offering her opinion on their authority as a matter of right; still less can she justly claim to herself the source of that authority."

A delicious humor is found in many passages. I wish to give but a single

example from De Morgan. "German intellect is an excellent thing, but when a German product is presented it must be analyzed. Most probably it is a combination of intellect (I) and tobacco-smoke (T). Certainly I_3T_1, and I_2T_1 occur; but I_1T_3 is more common, and I_2T_{15} and I_1T_{20} occur. In many cases metaphysics (M) occurs and I hold that $I_aT_bM_c$ never occurs without $b+c>2a$.

N.B.—Be careful, in analyzing the compounds of the three, not to confound T and M, which are strongly suspected to be isomorphic. Thus, $I_1T_3M_3$ may easily be confounded with I_1T_6. As far as I dare say anything, those who have placed *Hegel, Fichte*, etc., in the rank of the extenders of *Kant* have imagined T and M to be identical."

I have quoted freely from the volume under review, hoping thereby to convey something of the atmosphere that pervades it; but I realize fully that no limited number of quotations can adequately suggest the wealth and variety of ideas that it presents to the reader.

The author was very happy in the selection of material in this pioneer work, and has rendered a service to mathematical and non-mathematical readers.

The value of the work is greatly enhanced by the author's success in giving exact references to the quotations used, for in this way the Memorabilia becomes a guide to a much larger range of material "pertaining to mathematics, by poets, philosophers, statesmen, scientists, and mathematicians."

An excellent cross reference index of some 700 topics makes the material gathered very accessible.

GEO. N. BAUER.

PROBLEMS AND SOLUTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

ALGEBRA.

When this issue was made up solutions of 412 to 421 had been received. A solution of 406 is desired.

422. Proposed by W. D. CAIRNS, Oberlin College.

Find a solution of the equation $x^{x\sqrt{x}} = (x\sqrt{x})^x$.
(Adapted from Godfrey & Siddon's *Elementary Algebra.*)

423. Proposed by ELBERT H. CLARKE, Purdue University.

Show that the following formula is true for all positive integral values of k. The parenthetical symbols are defined as being the binomial coefficients. $\binom{k+1}{0} = 1$, by definition.

$$k^k\binom{k+1}{0} - (k-1)^k\binom{k+1}{1} + (k-2)^k\binom{k+1}{2} + \cdots + (-1)^{s-1}(k-s+1)^k\binom{k+1}{s-1} + \cdots + (-1)^k\binom{k+1}{k-1} = 1.$$

GEOMETRY.

When this issue was made up no solutions of 444, 446, 449 had been received. Please give attention to these.

451. Proposed by CLIFFORD N. MILLS, South Dakota State College.

Determine the sides of an isosceles triangle of given area, having given that the sum of its sides is equal to the sum of its base and altitude.

CALCULUS.

When this issue was made up no solutions of 348, 353, 354, 360, 363 had been received.

372. Proposed by V. M. SPUNAR, Chicago, Ill.

Find the condition that the equation:

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\left(1+\frac{a^2}{x^2}\right)y=0$$

should have one solution expressible in integral powers of x; and show that when this condition is satisfied, every other solution of the equation possesses a logarithmic infinity at the origin.

373. Proposed by C. N. SCHMALL, New York City, N. Y.

In the *Encyclopaedia Britannica* article on "Capillary Action" (Vol. 5, p. 268, 11th ed.) it is shown that $1/R_1 + 1/R_2 = p/T$, in the case of a soap bubble, where R_1, R_2 are the radii of curvature of a meridian section and a normal section, respectively, of the bubble; p, the difference of air-pressure; T, the energy per unit area of the film. Employing the principle that the soap bubble tends to assume a form such that the area of its surface is a *minimum* for a *given volume* of air, show by the Calculus of Variations that $1/R_1 + 1/R_2 = k$, where k is a constant.

MECHANICS.

Solutions of 286, 287, 288, 290, 291, 298, 299, 300 are desired.

300. Proposed by V. M. SPUNAR, Chicago, Ill.

A helical spring is composed of 20 turns of steel wire .258″ in diameter, the diameter of the coil being 3″. If the spring is compressed by a force of 50 lb., what is the maximum stress in the spring, its axial compression, and its resilience?

SOLUTIONS OF PROBLEMS.

ALGEBRA.

410. Proposed by C. N. SCHMALL, New York City.

Solve the simultaneous equations,

$$x^2+xy+y^2=a,$$
$$x^4+x^2y^2+y^4=b.$$

SOLUTION BY HORACE OLSON, Chicago, Ill.

The equations may be written

$$x^2+xy+y^2=a, \tag{1}$$

$$(x^2+xy+y^2)(x^2-xy+y^2)=b. \tag{2}$$

Whence

$$x^2-xy+y^2=b/a. \tag{3}$$

From (1) and (3),

$$x^2 + y^2 = (a^2 + b)/2a \quad \text{and} \quad 2xy = (2a^2 - b)/2a.$$

From these equations

$$x + y = \pm \sqrt{(3a^2 - b)/2a} \qquad \text{and} \qquad x - y = \pm \sqrt{(3b - 2a)/2a}.$$

Whence

$$x = \frac{1}{4a}[\pm \sqrt{6a^3 - 2ab} \pm \sqrt{6ab - 2a^3}],$$

$$y = \frac{1}{4a}[\pm \sqrt{6a^3 - 2ab} \mp \sqrt{6ab - 2a^3}].$$

Solved similarly by G. W. Hartwell, Paul Capron, Bernard Kramer, W. C. Eells, Clifford N. Mills, J. L. Riley, G. R. Mirick, H. C. Feemster, C. E. Githens, A. A. Nauer, Elmer Schuyler, and the Proposer.

Editorial Note.

These solutions are all conventional and do not state clearly,

(1) Which x's go with which y's. The four solutions obtained are, as to signs of the radicals,

$$\begin{array}{ll} x & ++ \mid +- \mid -+ \mid -- \\ y & +- \mid ++ \mid -- \mid -+ \end{array}$$

(2) Necessary and sufficient conditions for real roots.

They are

$$a > 0, \quad b > 0, \quad 9a^2 > 3b > a^2.$$

(3) The number of solutions is not that demanded by Bezout's theorem. There is no way of being sure that all the real solutions have been found except by producing the complete set of eight solutions and discarding those not acceptable. In this case the loci (1) and (2) have ordinary contact at each of the complex points at infinity in the directions whose slopes are ω and ω^2 (cube roots of unity). This accounts for the other four points and clinches the argument as to the real intersections.

GEOMETRY.

421. Proposed by R. P. Baker, University of Iowa.

Assuming the details of the proof of the existence of a sphere inscribed in a tetrahedron as usual in the texts, give an intuitional proof that there are in general eight spheres each touching the four faces, but for the regular tetrahedron only five. How many special types are there?

Solution by J. W. Clawson, Collegeville, Pa.

Call the planes passing through the six edges and bisecting the dihedral angles internally and externally respectively a, a'; b, b'; c, c'; d, d'; e, e'; f, f'. Call the planes BCD, CDA, DAB, and ABC I, II, III, and IV respectively (fig. 1).

A point will be said to be on the positive side of a face if it is on the same side of that face as a point inside the tetrahedron; if not it will be said to be on the negative side.

Now every point in a is equidistant from III and IV with same signs; every point in a' is equidistant from III and IV with opposite signs. So every point in b is equidistant from II and IV with same signs; every point n b' equidistant from II, IV with opposite signs. Similarly c and c' with respect to II and III;

d and d' with respect to I and IV; e and e' with respect to I and II; f and f' with respect to I and III.

It is easily proved, then, that a, b, c, d, e, f meet at O, a point equidistant from all faces and having the same sign with respect to all, i. e., the center of the inscribed sphere. Similarly that a, b, c, d', e', f' meet at E_7 a point equidistant from all four faces, the distances from II, III, IV having the same sign and that from I the opposite sign, i. e., the center of an escribed sphere touching the face I on the opposite side to the inscribed sphere. Similarly a, d, f, b', c', e' determine

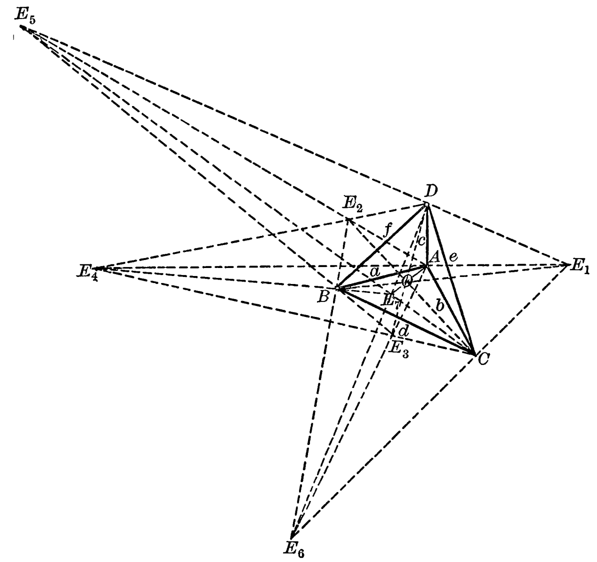

FIG. 1.

E_1, the center of the escribed sphere touching the face II opposite the inscribed sphere; b, d, e, a', c', f' determine E_2, the center of the escribed sphere touching face III opposite the inscribed sphere; c, e, f, a', b', d' determine E_3, the center of the escribed sphere touching the face IV opposite the inscribed sphere.

Again a, e, b', c', d', f' determine a point E_4, equidistant from all four faces, the distances from I and II having the same sign; and the distances from III and IV having the other sign. Similarly b, f, a', c', d', e' determine E_5, whose distances from I and III have the same sign and its distances from II and IV the other sign. And c, d, a', b', e', f' determine E_6, whose distances from I and IV have the same sign and its distances from II and III the other sign.

This gives us seven escribed spheres in all.

In the case where the tetrahedron is regular, three of these ex-centers, the three last named, E_4, E_5, E_6, are at infinity. To prove this, consider fig. 2. Bisect CD at F. The plane of ΔABF is perpendicular to edge CD and $\angle AFB$ is the plane angle of the dihedral angle CD. In ΔABF, if $AB = l$, $AF = BF = l\sqrt{3}/2$. Hence $\sin F/2 = 1/\sqrt{3} = \sin \angle AFI$. Draw planes bisecting AC,

CD, DA externally (planes b′, c', e'). These meet at E_1, forming a tetrahedron $ACDE_1$. Consider the vertex A. Face angle $CAD = 60°$, dihedral angle $AC = \cos^{-1} 1/\sqrt{3}$, dihedral angle $AD = \cos^{-1} 1/\sqrt{3}$. By spherical trigonometry, law of cosines, we find dihedral angle $AE_1 = 90^c$. Similarly, considering vertices C and D, we find dihedral angles CE_1, DE_1 to be 90°. Hence $AE_1 \perp E_1C, E_1D$; $AE_1 \perp$ plane of CE_1D; $AE_1 \perp E_1F$. But IF is $\perp E_1F$, whence AE_1 is $\parallel$ to IF.

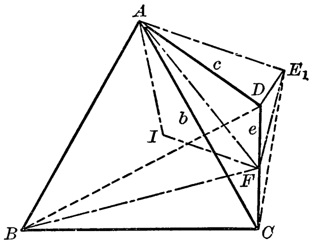

FIG 2.

Hence the line of intersection of a, b', and c' is parallel to the plane e and meets it only at infinity. So the point E_4 is at infinity. Similarly it may be shown that E_5 and E_6 are at infinity in this case.

This is the limiting case. In general AE_1 meets IF either on AE_1 produced or on E_1A produced.

Returning to fig. 1, A, E_1, E_4; A, E_2, E_5; A, E_3, E_6 are collinear. Also B, E_2, E_6; B, E_3, E_5; B, E_7, E_4; C, E_1, E_6; C, E_3, E_4; C, E_7, E_5; D, E_1, E_5; D, E_2, E_4; D, E_7, E_6.

Now if A, E_1, E_4 are collinear in that order (see fig. for Geom. 417), it is evident that the orders E_2, D, E_4; E_3, C, E_4; B, E_7, E_4; are determined. If A, E_2, E_5 are in that order, E_3, B, E_5; E_1, D, E_5; C, E_7, E_5 are determined orders. If A, E_3, E_6 are in that order, E_1, C, E_6; E_2, B, E_6; D, E_7, E_6 are determined orders.

If, however, E_1, A, E_4 be the order (see fig. 1), the orders D, E_2, E_4; C, E_3, E_4; E_7, B, E_4 obtain. And similarly with E_2, A, E_5 and E_3, A, E_6.

In special cases one, two, or three of the points E_4, E_5, E_6 may be at infinity. In that case the corresponding outer escribed sphere or spheres would not exist. We may therefore have four, five, six, or seven escribed spheres to a tetrahedron.

There are *two* types. (1) One of the vertices may be outside all three of the 'edgal' compartments in which the outer escribed spheres lie. In this case we may think of that vertex as at the top of the tetrahedron. The inscribed and three of the inner escribed spheres will then be above the level of the corresponding base, while the fourth inner and the three outer escribed spheres will be below that base. (2) One of the vertices may be at a corner of each of the three 'edgal' compartments in which the outer escribed spheres lie. In this case the three outer escribed spheres are above that vertex, the inscribed and three of the inner escribed spheres lie at a level between the vertex and its base and one of the inner escribed spheres only is below that base. It will be found

that if all seven escribed spheres are present the configuration can always be placed in one of these two types.

If one, two, or three of the outer spheres are lacking it will be found that we can place the configuration in *either* of the two types named above according as we consider the centers of the spheres which are at infinity to be *above* or *below* the vertex in question. These special cases occupy a limiting position between the two main types.

It may be pointed out that each vertex of the tetrahedron is collinear with four pairs of the eight points I, E_1, E_2, E_3, E_7, E_4, E_5, E_6. Fig. 1 shows the second type of tetrahedron; the figure that accompanies Geom. 417 shows the first type.

Editorial Note.—The discussion, scarcely intuitional, is printed for the division of the general case (7 finite ex-centers) into two sub-types. The intuitional view-point for the division into types with 4, 5, 6, 7 escribed spheres desired by the proposer is afforded by considering the angles at opposite pairs of edges. If equal the corresponding ex-center is at infinity, if unequal the ex-center is in the compartment of greater angle.

436. Proposed by A. J. KEMPNER, University of Illinois.

Given in a plane two similar curves arbitrarily situated, except that they shall possess the same sense of direction (which, of course, does not mean that they shall be similarly located). Let corresponding points on both curves be joined by straight lines, and let all of these straight lines be divided in the same ratio $\lambda : 1$, λ being any real number. Prove that the points of division all lie on a curve similar to the two given curves except when they all happen to fall together.

SOLUTION BY H. T. BIGELOW, Lafayette, Indiana.

Let the parametric equations of one of the curves be

$$x_1 = f(t), \qquad y_1 = \varphi(t). \tag{1}$$

The second curve, by reason of the similarity, is derivable from the first by an expansion from the origin, a rotation about the origin, and a translation. Its equation is, therefore,

$$\begin{aligned} x_2 &= a + k \cos \vartheta \cdot f(t) - k \sin \vartheta \cdot \varphi(t), \\ y_2 &= b + k \sin \vartheta \cdot f(t) + k \cos \vartheta \cdot \varphi(t), \end{aligned} \tag{2}$$

and corresponding points on the two curves (1) and (2) are given by equal values of t.

The corresponding point on the third curve is given by the equations

$$x_3 = \frac{x_1 + \lambda x_2}{1 + \lambda}, \qquad y_3 = \frac{y_1 + \lambda y_2}{1 + \lambda};$$

and

$$x_3 = \frac{\lambda a}{1 + \lambda} + \frac{(1 + \lambda k \cos \vartheta)}{1 + \lambda} f(t) - \frac{\lambda k \sin \vartheta}{1 + \lambda} \varphi(t),$$

$$y_3 = \frac{\lambda b}{1 + \lambda} + \frac{\lambda k \sin \vartheta}{1 + \lambda} f(t) + \frac{1 + \lambda k \cos \lambda}{1 + \lambda} \varphi(t).$$

If now we set $a' = \lambda a/(1 + \lambda)$ and $b' = \lambda b/(1 + \lambda)$ and choose k' and ϑ' so as to

make $k' \cos \vartheta' = (1 + \lambda k \cos \vartheta)/(1 + \lambda)$, $k' \sin \vartheta' = \lambda k \sin \vartheta/(1 + \lambda)$, the equations of the third curve become

$$\begin{aligned} x_3 &= a' + k' \cos \vartheta' \cdot f(t) - k' \sin \vartheta' \cdot \varphi(t), \\ y_3 &= b' + k' \sin \vartheta' \cdot f(t) + k' \cos \vartheta' \cdot \varphi(t). \end{aligned} \tag{3}$$

These equations are of the form (2) and consequently represent a curve similar to (1) and (2).

437. Proposed by J. BROOKS SMITH, Hampden Sidney, Va.

Let D, E, F be three arbitrary points taken on the sides of a triangle ABC. If Δ and Δ' be the areas of the triangles ABC and DEF, show that

$$\frac{\Delta'}{\Delta} = \frac{AF \cdot BD \cdot CE + AE \cdot CD \cdot BF}{abc},$$

the sign of each factor being determined as follows: Each segment adjacent to one of the vertices of the triangle ABC is to be regarded as positive or negative according as it is drawn towards or from the other vertex of the side containing the segment.

SOLUTION BY PAUL CAPRON, Annapolis, Md.

Let $AF = m$, $AE = q$, $BF = r$, $BD = k$, $CD = p$, $CE = l$; and let the areas of the triangles be $AFE = \Delta_1$, $BDF = \Delta_2$, $CED = \Delta_3$. Then, a, b, c being positive we have, in accordance with the given convention of signs, and the further convention that the area of a Δ shall be positive if the cyclic arrangement of its vertices is contra-clockwise, negative otherwise:

$$\frac{\Delta_1}{\Delta} = \frac{mq}{bc}, \qquad \frac{\Delta_2}{\Delta} = \frac{kr}{ca}, \qquad \frac{\Delta_3}{\Delta} = \frac{lp}{ab};$$

$$\frac{\Delta - \Delta_1 - \Delta_2 - \Delta_3}{\Delta} = \frac{\Delta'}{\Delta} = \frac{abc - amq - bkr - clp}{abc}.$$

Substituting $a = k + p$, $b = l + q$, $c = m + r$, we have

$$\frac{\Delta'}{\Delta} = \frac{klm + pqr}{abc} = \frac{AF \cdot BD \cdot CE + AE \cdot CD \cdot BF}{abc}.$$

Also solved by S. W. REAVES, H. C. FEEMSTER, C. N. SCHMALL, HORACE OLSON, and the PROPOSER.

438. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

By means of the theorem that the product of the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the two products of pairs of opposite sides, obtain the usual formulas for $\sin(\alpha \pm \beta)$ and $\cos(\alpha \pm \beta)$ in terms of $\sin \alpha$, $\sin \beta$, $\cos \alpha$, $\cos \beta$.

SOLUTION BY A. M. HARDING, University of Arkansas.

Suppose PR is a diameter and $\angle SPR = \alpha$ and $\angle RPQ = \beta$. (Fig. 1.) Then

$$PR \cdot SQ = PS \cdot RQ + PQ \cdot SR$$

or

$$2r \cdot 2r \sin(\alpha + \beta) = 2r \cos \alpha \cdot 2r \sin \beta + 2r \cos \beta \cdot 2r \sin \alpha.$$

Hence,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Suppose PQ is a diameter and $\angle SPQ = \alpha$ and $\angle RPQ = \beta$. (Fig. 2.) Then

$$PR \cdot QS = PS \cdot RQ + PQ \cdot SR$$

or

$$2r\cos\beta \cdot 2r\sin\alpha = 2r\cos\alpha \cdot 2r\sin\beta + 2r \cdot 2r\sin(\alpha-\beta).$$

Hence,

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

Suppose PQ is a diameter and $\angle RPQ = \alpha$ and $\angle SQP = \beta$. (Fig. 3.)

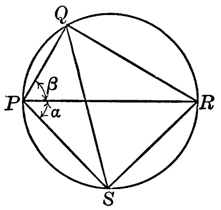

FIG. 1.

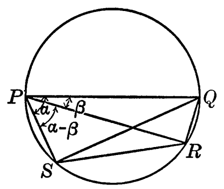

FIG. 2.

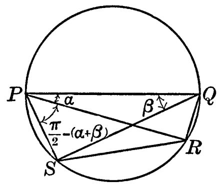

FIG. 3.

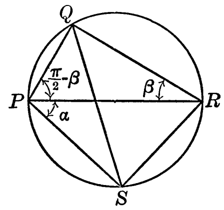

FIG. 4.

Then

$$PR \cdot SQ = PS \cdot RQ + PQ \cdot SR$$

or

$$2r\cos\alpha \cdot 2r\cos\beta = 2r\sin\beta \cdot 2r\sin\alpha + 2r \cdot 2r\sin[\pi/2 - (\alpha+\beta)].$$

Hence,

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Suppose PR is a diameter and $\angle SPR = \alpha$ and $\angle PRQ = \beta$. (Fig. 4.) Then

$$PR \cdot SQ = PS \cdot RQ + PQ \cdot SR$$

or

$$2r \cdot 2r\sin[\pi/2 + (\alpha-\beta)] = 2r\cos\alpha \cdot 2r\cos\beta + 2r\sin\beta \cdot 2r\sin\alpha.$$

Hence,

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Also solved by C. N. SCHMALL.

439. Proposed by CHARLES N. SCHMALL, New York City.

Show that the areas of any two triangles circumscribed about the same circle are in the same ratio as their perimeters.

SOLUTION BY A. L. MCCARTY, Cape Girardeau, Mo.

Let the radius of the circle be r and the sides of the circumscribed triangles be a, b, c and d, e, f respectively.

Now it is evident that the area of the first triangle is $\frac{1}{2}r(a+b+c)$ and the area of the second is $\frac{1}{2}r(d+e+f)$. Hence, the two triangles are to each other as their perimeters.

Also solved by HORACE OLSON, J. L. RILEY, H. C. FEEMSTER, C. E. FLANAGAN, CLIFFORD N. MILLS, C. E. GITHENS, GEO. W. HARTWELL, ELMER SCHUYLER, WALTER C. EELLS, and A. M. HARDING.

MECHANICS.

288. Proposed by C. E. HORNE, Westminister College, Colorado.

Show that the tangential velocity of a projectile at any point of its path is equal to the velocity it would have acquired in falling, under the influence of gravitation alonè, from the directrix to the point in question.

SOLUTION BY A. M. HARDING, University of Arkansas.

The equation of the path in parametric form is

$$x = u \cos \alpha \cdot t, \qquad y = u \sin \alpha \cdot t - \tfrac{1}{2}gt^2,$$

where u is the initial velocity and α is the angle of projection.

Then

$$dx = u \cos \alpha \, dt, \qquad dy = (u \sin \alpha - gt)dt,$$

$$ds^2 = dx^2 + dy^2 = (u^2 - 2ug \sin \alpha \cdot t + g^2t^2)dt^2 = (u^2 - 2gy)dt^2.$$

Now it can be easily shown that the distance from the directrix to the X-axis is given by $d = u^2/2g$. Hence $ds^2 = (2gd - 2gy)dt^2$.

Whence

$$\frac{ds}{dt} = \text{velocity} = \sqrt{2g(d - y)}.$$

Hence, the velocity at any point is equal to the velocity it would have acquired in falling from the directrix.

Also solved by ELIJAH SWIFT, CLIFFORD N. MILLS, HORACE OLSON, J. W. CLAWSON.

NUMBER THEORY.

207. Proposed by A. J. KEMPNER, University of Illinois.

There are 80 positive integers < 100 containing no figure 9 against 19 containing at least one figure 9. (For integers < 1000 the numbers are 728 and 271 respectively.) One might be led to believe that for every positive integer M the number N_1 of positive integers $< M$ containing no figure 9 is always greater than the number N_2 of positive integers $< M$ containing at least one figure 9.

To prove: $\lim (N_1/N_2) = 0$ for $M \doteq \infty$.

SOLUTION BY LOUIS O'SHAUGHNESSY, University of Pennsylvania.

Let N_2 in every case represent the number of positive integers from 1 to M inclusive, which contain the figure 9 at least once; while N_1 represents the number of positive integers from 1 to M inclusive, which do not contain the figure 9. Then for

$$M = 10, \qquad N_2 = 1,$$

$$M = 10^2, \qquad N_2 = 9 \times 1 + 10, \text{ or } 9 + 10.$$

$$M = 10^3, \qquad N_2 = [9 \times 1 + 10]9 + 10^2, \text{ or } 9^2 + 9 \cdot 10 + 10^2.$$

$M = 10^4, \quad N_2 = \{[9 \times 1 + 10]9 + 10^2\}9 + 10^3$, or $9^3 + 9^2 \cdot 10 + 9 \cdot 10^2 + 10^3$.

$M = 10^5, \quad N_2 = 9^4 + 9^3 \cdot 10 + 9^2 \cdot 10^2 + 9 \cdot 10^3 + 10^4$.

. . . .

$M = 10^t, \quad N_2 = 9^{t-1} + 9^{t-2} \cdot 10 + 9^{t-3}10^2 + \cdots + 9 \cdot 10^{t-2} + 10^{t-1}$.

$M = 10^{t+1}, \quad N_2 = 9^t + 9^{t-1} \cdot 10 + 9^{t-2}10^2 + \cdots + 9^2 \cdot 10^{t-2} + 9 \cdot 10^{t-1} + 10^t$.

In general, for $M = 10^t$, $N_2 = (10^t - 9^t)/(10 - 9) = 10^t - 9^t$; $N_1 = 9^t$.

$$\operatorname*{Lim}_{t \doteq \infty} \frac{N_1}{N_2} = \lim_{t \doteq \infty} \left[\frac{9^t}{10^t - 9^t} \right] = 0.$$

For the more general values of M such as $a_0 10^s + a_1 10^{s-1} + \cdots + a_{s-1}10 + a_s$, wherein none of the a's is equal to 9,

$$N_2 = a_0(10^s - 9^s) + a_1(10^{s-1} - 9^{s-1}) \cdots + a_{s-1}(10 - 9),$$

while

$$N_1 = a_0 9^s + a_1 9^{s-1} + \cdots + a_{s-1}9 + a_s.$$

Hence,

$$\lim_{s \doteq \infty} \text{ of } \frac{N_1}{N_2} = 0.$$

If a_3 should be the first a which is 9, then

$$N_2 = a_0(10^s - 9^s) + a_1(10^{s-1} - 9^{s-1}) + a_2(10^{s-2} - 9^{s-2}) + a_3(10^{s-3} - 9^{s-3}) + a_4 10^{s-4} + \cdots + a_{s-1}10 + a_s + 1,$$

while

$$N_1 = a_0 9^s + a_1 9^{s-1} + a_2 9^{s-2} + a_3 9^{s-3} - 1.$$

Hence,

$$\lim_{s \doteq \infty} \frac{N_1}{N_2} = 0.$$

With very slight changes the formula may be made to apply to the corresponding cases for 0, 1, 2, $\cdots$ 8.

MISCELLANEOUS QUESTIONS.

EDITED BY R. D. CARMICHAEL.

Editorial Note.—Several questions of interest are still outstanding. Question **4** is important for teaching analytic geometry (see December, 1913, issue). All efforts to increase the number of concrete applications available are to be commended.

Question **6** (February, 1914, issue) is of great significance in view of the present strong tendency toward increasing the vocational training in the schools.

Question **8** (March, 1914, issue) has an important bearing upon the relation of the colleges to the schools of education. Should not professional training for

teachers include practice teaching? In some cases only *observation* is expected, in others the practice teachers are put in actual charge of classes for a semester and given full responsibility under supervision. It has long been recognized that the training of teachers for elementary schools should include practice teaching; there is now at least one state college for the preparation of high school teachers where the responsible practice teaching is an important feature; why should not some sort of practice teaching be required of advanced students who are preparing for college positions?

Question **9** (March issue), so far as it pertains to coördinated courses in college mathematics, has just acquired a fresh interest in the new book by Professor Slichter, of the University of Wisconsin, entitled "Elementary Mathematical Analysis," which is designed to provide work equivalent to the traditional courses in trigonometry, college algebra, and analytic geometry. This book will be reviewed in the MONTHLY at an early date. Meanwhile, contributions concerning the general question of coördinated collegiate mathematics are desired.

Questions **11** to **15** (issues of May, June, September, and October) are of more limited interest, but nevertheless represent precisely the attitude of mind that this department is striving to foster, namely, a real question about a real difficulty encountered, or a genuine desire to gain further light upon important matters connected with the teacher's work.

Question **13** is possibly an exception to this last statement in that it seems to be an inquiry as to what is going on in the world outside rather than an expression of inner conflict of opinion. But, even so, it is a most worthy question. The teacher must look to the world outside for inspiration and courage to escape from submersion in the quiet waters of the daily routine. Professor Lehmer in California was surprised to find that Professor Bussey in Minnesota did not know about his course in synthetic projective geometry which he had been giving to junior college students for ten years. But how can we know about these things unless we *communicate* them to others? And how can we communicate them without a *medium* of communication? This is the mission of the MONTHLY and especially of the department of "Miscellaneous Questions." If every one who has something good which he has been concealing will come forward and let his light shine for the benefit of others, there will be a general awakening all along the line.

NEW QUESTIONS.

16. To what extent should a first course in geometry be made to appeal to a student's intuition? Should the subject be presented in a manner so as to depend to the greatest possible degree upon his previous experience, or is it desirable to attempt to make it more abstract and formal?

17. In analytic geometry, simplicity and directness are gained by making the condition for the collinearity of three points and the equation of the straight line depend upon the determinant formula for the area of a triangle. Similar advantages are gained by making the condition of complanarity of four points and the equation of the plane depend upon the determinant formula for the volume of a tetrahedron. The former is given in the texts. Why should not the latter be given? A uniform method of developing these two determinants is desired from some contributor.

18. In view of the present pressure for saving time and gaining efficiency, what are the most important sources of economy in the mathematical courses of the high school and the first two years in college?

NOTES AND NEWS.

Edited by W. DeW. Cairns.

The mathematical physical society of Kazan, Russia, offers the Lobatchewski prize of 500 roubles for productions relative to non-euclidean geometry, these to be submitted by Nov. 4, 1914.

Professor William J. Milne, Ph.D., LL.D., president of the New York State Normal College, Albany, N. Y., and author of many school texts in arithmetic, algebra and geometry, died September 4 at Bethlehem, N. H.

A second edition of Professor E. L. Thorndike's "Introduction to the Theory of Mental and Social Measurements" which was published in 1904 (Science Press, New York) has recently appeared.

In the May number of *Popular Science Monthly* occurs a paper by Professor William Marshall on "The theory of relativity and the new mechanics."

Professor W. F. Osgood now holds the Perkins professorship of mathematics at Harvard University, the chair formerly held by Professor W. E. Byerly.

Harvard University conferred the degree of doctor of science upon Professor W. C. Sabine at the commencement in June, 1914. He now holds the Hollis professorship of mathematics and natural philosophy at Harvard, the chair formerly held by Professor B. O. Peirce.

The thirty-fourth regular meeting of the Chicago Section of the American Mathematical Society will be held at the University of Chicago on Monday and Tuesday, December 28, 29, 1914, the first session opening at 10 o'clock a. m., in Room 32, Ryerson Physical Laboratory.

Professor R. C. Archibald has an article in the May *Bulletin of the American Mathematical Society* on "Time as a fourth dimension."

A warm tribute by Professor John Trowbridge to the memory of the late Professor B. O. Peirce, of Harvard University, will be of great interest to the many American physicists and mathematicians who have studied with Professor Peirce. This appears in a recent number of the *Harvard Graduates' Magazine*.

Mr. R. A. Johnson, instructor in mathematics in Adelbert College, Western Reserve University, offers for the first semester of the year 1914–15 a teachers' course in elementary geometry. The meetings of the course are held at four o'clock in the afternoon, an hour which makes possible the attendance of teachers in the Cleveland public schools.

At the tercentenary celebration at the University of Groningen, the degree of doctor of science was conferred upon Professor E. B. VAN VLECK of the University of Wisconsin.

Professor F. ANDEREGG, of Oberlin College, has returned after a year's leave of absence. While in London he submitted to an operation for appendicitis, from which he recovered speedily.

A reprint from "*The Merchistonian*" sent to the editor by Professor F. CAJORI from Edinburgh contains a careful historical description of the Tower of Merchiston, later Merchiston House, and a brief but entertaining sketch of the life of John Napier with the reproduction of two of his portraits.

W. E. EDINGTON, B.A., Indiana State Normal School, graduate student in the University of Chicago, 1912–13, and instructor in the University of Colorado, 1913–14, has been elected professor of mathematics in the University of New Mexico at Albuquerque.

Because considerable difficulty has been experienced now and then by American teachers in obtaining the reports of the Mathematical Association, we reprint the following notice from the *Mathematical Gazette*, the organ of the Association:

> The following Reports have been issued by the Association:—(i) "Reports on the Teaching of Elementary Mathematics," 1902–1908 (Geometry, Arithmetic and Algebra, Elementary Mechanics, Advanced School Mathematics, the Course required for Entrance Scholarships at the Universities, Mathematics in Preparatory Schools), price 6d. net; (ii) "Revised Report on the Teaching of Elementary Algebra and Numerical Trigonometry" (1911), price 3d. net; (iii) "Report on the Teaching of Mathematics in Preparatory Schools," price 3d. net; and (iv) "Report on the Correlation of Mathematical and Science Teaching," by a Joint Committee of the Mathematical Association and the Association of Public School Science Masters, price 6d. net; (v) A General Mathematical Syllabus for Non-Specialists in Public Schools, price 2d. net. These reports may be obtained from the Publishers of the *Gazette*, G. Bell and Sons, Ltd., Portugal St., Kingsway, London.

Among the articles contained in "The Mathematics Teacher" in the issues of March and June, 1914, the following may be mentioned as of interest to our readers: "Mathematics as a means to culture and discipline" (13 pages) by A. D. Yocum; "What mathematical knowledge and ability may reasonably be expected of the student entering college" (8 pages) by J. N. Hart; "Freshman algebra and the average freshman" (16 pages) by Ella D. Gray; and "Mathematics: the subject and the teacher" (12 pages) by F. C. Ferry.

The discussion of "A Kinetic Theory of Gravitation" by Charles F. Brush in *The Proceedings of the American Philosophical Society* for January–May, 1914, calls attention to a new way by which we may account for the immense forces in operation in the mutual attractions of astronomical bodies.

Today the science of mathematics is so vast and its development is so rapid that no one unaided could keep in touch with its growth in a single important field of considerable extent. Consequently it is necessary to have published reviews of the articles so that by means of these the individual worker will be able to determine quickly what are the recent additions to knowledge along the line in which he is especially interested. These reviews should meet two requirements which militate against each other, namely, they should be available promptly after the appearance of the article and they should be carefully prepared. The first of these requirements is met in *Revue Semestrielle des Publications Mathématiques*, an excellent and very prompt review of nearly all additions to mathematical literature; it does not appear to be used as much in this country as it might be. The second requirement is best met in the well-known *Jahrbuch über die Fortschritte der Mathematik.*

In recognition of the sixth international congress of mathematicians, to be held in Stockholm in 1916, King Gustave V has offered a gold medal bearing the likeness of Karl Weierstrass and a prize of 3,000 crowns, for any important discovery in the field of the theory of analytic functions. All manuscripts are to be submitted to *Acta mathematica* before October 31, 1915, the hundredth anniversary of the birth of Weierstrass.

The French Academy of Science offers the following prizes: (1) *Prix Bordin* for 1915 (3,000 francs), for making any noteworthy progress in the investigation of curves of constant torsion, and for determining if possible such of these curves as are algebraic, or at least such as are unicursal. (2) *Grand Prix des Sciences mathématiques* for 1916 (3,000 francs), for applying the methods of Henri Poincaré to the integration of linear algebraic differential equations, selected from the simplest. (3) *Prix Bordin* for 1917 (3,000 francs), for perfecting in any important way the arithmetical theory of non-quadratic forms. (4) *Prix Vaillant* for 1917 (4,000 francs), for determining and investigating all the surfaces which can be generated in two different ways by the displacement of an invariable curve.

Papers must be submitted to the Academy before the close of the year preceding the awarding of the various prizes.

Recent numbers of *Rendiconti del Circolo Matematico di Palermo* contain the following papers by Americans: Professor G. D. BIRKHOFF, "Note on the expansion problems of ordinary linear differential equations"; Professor J. EIESLAND, "On the algebraic curves of a tetrahedral complex and the surfaces conjugate to it"; Professors G. N. BAUER and H. L. SLOBIN, "Some transcendental curves and numbers"; Professor W. H. METZLER, "Rectangular arrays"; Dr. DUNHAM JACKSON, "A formula of trigonometric interpolation." This journal has published as a reprint the most important speeches and articles in memory of Henri Poincaré, together with a letter from Poincaré to the editor of *Rendiconti* on the subject of his last memoir, and a list of the memoirs of Poincaré which appeared in *Rendiconti* from 1888 to 1912.

The first regular meeting of the Association of Mathematics Teachers of New Jersey was held at Rutgers College, New Brunswick, N. J., on November 7th. Further notice of this meeting and of the purpose of this new organization will be given in a later issue of the MONTHLY.

The office of the American Mathematical Society, fitted out a year ago by Columbia University, was completely destroyed by a fire in University Hall on October 10, with loss of furniture, records, and stock of *Bulletin* and *Transactions* amounting to over $2000. The Society has duplicate stock of the *Transactions* and of the later years of the *Bulletin*. But the early volumes of the latter (as far as Vol. X) were completely lost. The Society will be glad to receive copies of these early volumes from its members and others who may be willing to donate them.

At the Pennsylvania State College, Dr. E. R. SMITH has been promoted to an associate professorship in mathematics and Dr. J. E. ROWE of Dartmouth College has been elected to an assistant professorship. It may be noted concerning the scope of work of this college that courses will be given during the fall term in theory of functions of real and complex variables, differential equations, functions defined by differential equations, statistical methods, Newtonian potential, and theory of numbers. Professor Stecker of this faculty gave one of the first courses in integral equations offered in the United States.

To pay proper tribute to the memory of Henri Poincaré and to link his name with an endowment for scientific purposes, a very representative international committee offers to the friends, colleagues and admirers of Poincaré the opportunity of sharing in a subscription which (1) shall furnish medals with the likeness of Poincaré, and (2) shall constitute a fund, the income from which is to be used by the Academy of Science to encourage younger scholars engaged in those branches of science whose development has been assured by his genius; namely, mathematical analysis, celestial mechanics, mathematical physics, philosophy of science. A bronze medal is given those subscribing from 25 to 50 francs, a silver medal for larger subscriptions. Subscriptions are to be sent to ERNEST LEBON, rue des Écoles, n° 4, Paris 5e.

The twenty-first summer meeting of the American Mathematical Society was held at Providence, R. I., on Tuesday and Wednesday, September 8, 9, 1914, as a part of the ceremonies connected with the celebration of the one-hundred and fiftieth anniversary of the founding of Brown University. There were fifty-two members present, including twelve from the middle west and far west, and thirty-three papers were presented during three half day sessions. The entertainment of the members given by President Faunce, Chancellor Chace, Professor Davis, and other members of the mathematical faculty of Brown University, was so generous and whole hearted that every one present will carry the most pleasant recollections of it for years to come. There was a luncheon,

an afternoon tea, a reception, a dinner, and last but not least a personally conducted tour about Newport under the guidance of Professors Richardson and Archibald, who as local members of the committee of arrangements were everywhere present and all things to all of us. President Faunce gave a notable address in his inimitable style at the dinner on Tuesday evening and Professor Carl Barus, of the department of physics in Brown University, gave an illuminating talk on "The relations of mathematics to physics." The mere enumeration of all these good things, not to mention the friendly discussions of the scientific papers presented and the general good fellowship always found on these occasions, should be sufficient to make every member of the Society resolve to attend the next meeting, which is to be held in San Francisco during the summer of 1915.

SPECIAL ANNOUNCEMENT.

As important evidence that the MONTHLY will maintain in 1915 the same high standard which has been set in the two years since its reorganization, we call attention to the following announcement:

Under the title " History of Zeno's Arguments on Motion," Professor Cajori will begin in the January, 1915, issue a series of articles which promise to be of unusual interest. Our present well-organized theory of limits, like every modern system of thought in all fields, has its roots far back in human history. Just as the ideas of negative and imaginary numbers struggled for recognition and formulation through hundreds of years, so the notion of a variable approaching a limit and all of the far-reaching consequences connected with the consideration of such a concept have gone through a process of slow development. Professor Cajori traces this development through a two-thousand-year struggle for light in which practically all of the great mathematicians and philosophers of the time were engaged. These articles will begin with the Volume XXII and will extend well through the year 1915. The editors are greatly pleased that we are able to offer such an attraction to the readers of the MONTHLY for the coming year. No teacher who has to deal with the theory of limits can afford to miss this latest product of Professor Cajori's historical researches.

As most subscriptions expire with the December issue, the renewal notices will be sent out in a few days, in order that all may have ample time to renew their subscriptions before the end of the year, and thus be sure of receiving the January issue containing the first instalment of Professor Cajori's articles.

The editors will greatly appreciate your prompt action in this matter, and they would suggest that you can do a real service to those in your acquaintance by calling their attention to this unique opportunity and sending their names to the Managing Editor for sample copies.

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THE THEOREM OF ROTATION IN ELEMENTARY MECHANICS.[1]

By EDWARD V. HUNTINGTON, Harvard University.

In order to discuss the motion of a rigid body in a plane, we need to know two things: (1) the translational motion of some one point of the body; and (2) the angular motion of some one line of the body.

Information in regard to the first point is obtained from the well known theorem on the motion of the center of mass.

Information in regard to the second point is obtained from a theorem which is variously known as the theorem of rotation, the theorem of moments, the theorem of angular acceleration, etc.

This theorem of rotation is usually written in the form

$$T = I\dot{\omega},$$

where $T =$ the torque, or turning moment, of the external forces about some suitably chosen point Q, $I =$ the moment of inertia of the body about that point, and $\dot{\omega} =$ the angular acceleration of the body.

In order to bring out the "dimensions" of the quantities involved, the equation may be written in the more explicit form

$$\Sigma Fp = \frac{W}{g} k^2 \frac{d\omega}{dt},$$

where $\Sigma Fp =$ the sum of the moments of the external forces about the given point Q, $k =$ the radius of gyration of the body about that point, and $d\omega/dt =$ the angular acceleration of the body at the instant in question.

This equation may be called the equation obtained by "taking moments about the point Q," and the question at once arises: for what choices of the point Q will the equation be valid? In other words, about what points is it legitimate to "take moments"?

Surprising as it may seem, the elementary text books are not in agreement in their answer to this question.

[1] Presented to the American Mathematical Society at the Summer meeting, September 8, 1914.

All the books do indeed agree that the equation is true if Q is a fixed point, that is, a point fixed in the body and fixed in space; and also if Q is the center of mass. That is, it is always legitimate to take moments about a fixed axis, or about the center of mass.

But many books go further and state that the equation is true when Q is the instantaneous center, and solve many problems by taking moments about the instantaneous center. For example we mention only the familiar problem of the motion of a cylinder rolling down an inclined plane, which is often solved by taking moments about the point of contact.

And finally, some books go so far as to state that the equation is true no matter what point may be taken as the point Q.[1]

In view of these discrepancies among current text books, it seems worth while to review the development of this important equation, and to state its precise scope and limitations.

We shall see that the equation is not true, in general, when the point Q is chosen at random, and that it is not even true, in general, when Q is the instantaneous center; on the other hand, we shall see that it is true for certain other choices of the point Q, which are not usually mentioned.

For the sake of clearness, we shall distinguish two distinct theorems, the first of which may be called the theorem of moments about a point fixed in the body, and the second, the theorem of moments about a point fixed in the plane. The proof of these theorems, which is ordinarily made to depend on the general theory of angular momentum (couched in a series of rather long and forbidding equations), is here developed from much more elementary considerations, and by steps which, it is believed, can be readily recalled to mind by any student who has once been through the reasoning.

Theorem I. *Theorem of moments about a point fixed in the body.* If a rigid body W is free to move in a plane, under the action of any external forces, and if Q is any *suitably chosen* point of the body (see below), then, at any instant,

[1] Of the recent elementary text books which have come under my notice, the following widely used books are among those which take the extreme view that the equation of rotation is always true, for any point Q: SMITH and LONGLEY, *Theoretical Mechanics*, p. 236; H. M. DADOURIAN, *Analytical Mechanics*, 1913, § 184 and § 188; FULLER and JOHNSTON, *Applied Mechanics*, vol. 1, 1913, pp. 308–309. The statement of Fuller and Johnston is particularly explicit on this point: "When a body has imparted to it a plane motion of rotation about *a moving or a fixed axis*, the motion may be resolved into a motion of rotation relative to an axis perpendicular to the plane of motion through *any point A*, with an angular velocity ω equal to that of the body about the axis of rotation, and a motion of translation with a linear velocity V_a equal to that of the point A. . . . The resultant of the system of forces required to impart to a rigid body the motion described above may be determined by finding the resultant of the system of forces required to impart the motion of rotation relative to the axis A, and the resultant of the system of forces required to impart the motion of translation, and combining the two resultants. *While the point A may be any point within the rotating body,* in most problems dealing with motion of this kind the simplest solution can be made by taking A at the center of gravity." (Italics are mine.) This is a particularly good example of the confusion which may result from speaking of the angular velocity of a body *about an axis,* when the axis is not fixed. The angular velocity of a body should be defined without reference to any axis of rotation, as simply the rate of change of the angle which a line fixed in the body makes with a line fixed in the plane.

$$\Sigma Fp = \frac{W}{g} k^2 \frac{d\omega}{dt},$$

where ΣFp = the sum of the moments of the external forces about the point Q (in the positive direction of rotation) at the instant in question, and $d\omega/dt$ = the angular acceleration of the body at that instant (θ being measured in radians, in the positive direction of rotation, and $\omega = d\theta/dt$); while k = the radius of gyration of the body about the point Q (that is, about an axis through Q perpendicular to the plane of the motion).

Here by a "suitably chosen" point Q we mean any point Q of the body such that (1) Q is fixed in the plane; or (2) Q is moving with uniform velocity in a straight line; or (3) Q is the center of mass; or (4) Q has an acceleration whose direction, at the instant in question, passes through the center of mass; and the equation above will be valid when and only when Q has some one of these four properties.

Proof of Case 1. Entirely elementary proofs for the case of rotation about a fixed axis are readily accessible and need not be repeated here. We shall mention only that this case can be conveniently recalled to mind when stated in the following form: If a rigid body is free to rotate about a fixed axis, under the action of any turning moment, the angular acceleration at any instant is the same as if all the material of the body were concentrated in a single particle at a distance k from the axis, where k is the radius of gyration of the body about the axis.

Proof of Case 2. First, regard the given body as a system of free particles acted on by definite external and internal forces, under given initial conditions (the internal forces being just sufficient to keep the system rigid); and let V_x, V_y be the initial values of the velocity components of the given point Q, referred to fixed axes in the plane.

Now imagine the initial conditions so changed that the initial velocity components of each and every particle are diminished by the amounts V_x and V_y, all the forces remaining the same as before. In the new situation thus imagined, it is easy to see: (1) that the system will still move as a rigid body, and (2) that its angular velocity and angular acceleration will remain unchanged; but the point Q will be reduced to rest.

Under these new conditions, therefore, we may apply the theorem of case 1, thus obtaining the equation $\Sigma Fp = (W/g)k^2 d\omega/dt$, where all the letters have their original meanings. The truth of the equation is thus proved for case 2.

Proof of Case 4, which includes case 3. Regard the given body again as a system of free particles, as in case 2, and let a_x and a_y be the acceleration components of the given point Q, referred to fixed axes in the plane.

Now imagine impressed upon each and every particle w an additional force with components $-(w/g)a_x$ and $-(w/g)a_y$; this will clearly have the effect of diminishing the acceleration components of each particle by the amounts a_x and a_y. In the new situation thus imagined, it is easy to see: (1) that the system will still move as a rigid body, and (2) that its angular velocity and angular

acceleration will remain unchanged; but the point Q will be reduced to a state of uniform motion in a straight line.

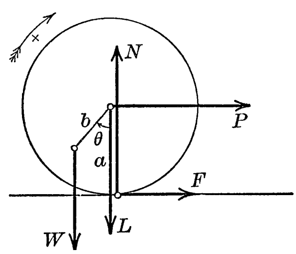
Under these new conditions, therefore, we may apply the theorem of case 2, and thus obtain the equation

$$\Sigma Fp + \Sigma f'p' = (W/g)k^2 d\omega/dt,$$

where ΣFp and ω have their original meanings, and $\Sigma f'p'$ = the sum of the moments about Q of the additional impressed forces.

Now it will be noticed that these additional forces form, at each instant, a set of parallel forces each of which is proportional to the mass of the particle on which it acts. Hence, by the definition of the center of mass, in finding the sum of the moments of these parallel forces, we may replace the entire set of forces by a single force F' acting at the center of mass. Furthermore, the direction of these parallel forces, and hence the direction of F', is opposite to the direction of the vector acceleration of the point Q; but, by the hypothesis of case 4, this direction lies along a line drawn through Q and the center of mass. Hence the moment of F' about Q will be zero; that is, $\Sigma f'p' = 0$, and the theorem is established.

It should be carefully noted that in the statement of theorem I the instantaneous center—that is, the point of no velocity—is not included in the list of "suitable" points Q; in other words, it is not legitimate to "take moments about the instantaneous center," unless this point happens to have also one of the other properties explicitly mentioned in the theorem.



To illustrate the danger of "taking moments about the instantaneous center," consider the following simple problem.

Suppose a carriage is moving with uniform velocity along a level road, and consider the motion of a wheel, of radius a, in which the center of mass does not coincide with the center of the wheel. The center of mass will then move in a trochoidal path, and the components of its acceleration will be $\ddot{x} = b\omega^2 \sin\theta$, $\ddot{y} = b\omega^2 \cos\theta$, where b is the distance from the hub to the center of mass, and ω the constant angular velocity of the wheel.

The forces acting on the wheel are (1) the weight W, acting downward at the center of mass; (2) the pressure of the bearings, which we may suppose resolved

into a forward component P, and a downward component L; and (3) the reaction of the road, which we may suppose resolved into a forward component F, and an upward component N. The equations of motion for the center of mass are therefore

$$P + F = (W/g)b\omega^2 \sin \theta, \tag{1}$$

$$N - W - L = (W/g)b\omega^2 \cos \theta. \tag{2}$$

Since the angular velocity is constant, $d\omega/dt = 0$, and the equation of rotation may be written as follows:
(a) about the center of mass:

$$Pb \cos \theta + (L - N)b \sin \theta - F(a - b \cos \theta) = 0; \tag{3a}$$

(b) about the hub, which is moving with uniform velocity in a straight line:

$$Fa + Wb \sin \theta = 0; \tag{3b}$$

(c) about the lowest point of the wheel, which is the instantaneous center:

$$Pa - Wb \sin \theta = 0. \tag{3c}$$

Of the last three equations, (3a) and (3b) are true, in accordance with theorem I, and from either of them, combined with (1) and (2), we find

$$Pa - Wb \sin \theta = \frac{W}{g} ab\, \omega^2 \sin \theta. \tag{4}$$

But this shows that equation (3c) is false. Hence in this case, wholly erroneous results would be obtained by taking moments about the instantaneous center.

If the center of mass had coincided with the hub, it would then indeed have been legitimate to take moments about the lowest point of the wheel, not, however, because this point is the instantaneous center, but because it is in this case a point whose vector acceleration at the instant in question passes through the center of mass (case 4). The use of case 4 and case 2 will often lead to simpler equations that could be obtained if we confined ourselves to the use of the more familiar cases 1 and 3.

Corollary I. In discussing the motion of a rigid body in a plane, any given set of external forces may be replaced by (1) a single force, acting at the center of mass, and (2) a couple, acting anywhere in the body. The components of the single force will be $(W/g)a_x$ and $(W/g)a_y$, where a_x and a_y are the components of the acceleration of the center of mass. The moment of the couple will be $(W/g)k^2 d\omega/dt$, where $d\omega/dt$ is the angular acceleration of the body, and k its radius of gyration about the center of mass.

The single force, acting at the center of mass, will have no effect on the angular velocity of the body; while the couple will have no effect on the motion of the center of mass.

It is clear, however, from theorem I, that a single force whose line of action does not pass through the center of mass will produce not only a change in the motion of the center of mass, but also a change in the angular velocity of the body.

Corollary II. In discussing the motion of a rigid body in a plane, the point of application of any force may be shifted at pleasure along the line of action of that force.

For the moment of the force about any point, and the components of the force along any axes will be unchanged.

This corollary, known as the principle of transmissibility of forces, is usually assumed as a fundamental principle of statics. But this assumption is not required for the proof of the theorems here stated.

Theorem II. *Theorem of moments about a point fixed in the plane.* If a rigid body W is free to move in a plane, under the action of any external forces, and if O is any point fixed in the plane, then, at any instant,

$$\Sigma Fp = \frac{W}{g}\frac{d(h\bar{v})}{dt} + \frac{W}{g}\bar{k}^2\frac{d\omega}{dt},$$

where ΣFp = the sum of the moments of the external forces about the point O (in the positive direction of rotation) at the instant in question, and

$h\bar{v}$ = the moment about O of the velocity $\bar{v}$ of the center of mass at that instant; while

$\bar{k}$ = the radius of gyration of the body *about its center of mass.*

Proof. Let C be the point of the plane through which the center of mass is passing at the instant in question, and let a_r and a_θ be the acceleration components of the center of mass along and perpendicular to the fixed axis OC at that instant.

By corollary I, the external forces acting on the body may be replaced by (1) a single force, acting at the center of mass, with components $(W/g)a_r$ and $(W/g)a_\theta$ along and perpendicular to OC; and (2) a couple, acting anywhere in the body, with a moment (about any point) equal to $(W/g)\bar{k}^2 d\omega/dt$. Hence the sum of the moments of the external forces about O reduces to

$$\Sigma Fp = (W/g)ra_\theta + (W/g)\bar{k}^2 d\omega/dt,$$

where $r = \overline{OC}$. But from the elementary kinematics of a particle in polar coordinates, $a_\theta = \dfrac{1}{r}\dfrac{d(r\bar{v}_\theta)}{dt}$, where $\bar{v}_\theta = \dfrac{rd\theta}{dt}$ = the velocity component of the center of mass in the direction perpendicular to OC. Also, $r\bar{v}_\theta = h\bar{v}$, where h is the perpendicular from O to the vector $\bar{v}$. Hence

$$\Sigma Fp = (W/g)d(h\bar{v})/dt + (W/g)\bar{k}^2 d\omega/dt,$$

which establishes the theorem.

This theorem II is by no means so simple or so important as theorem I. It is useful chiefly in problems in impact, especially in the derived form obtained by multiplying through by dt and integrating from $t = t_0$ to $t = t_1$. In this derived form the equation is known as the equation of impulse and momentum. But neither of these concepts is required for the proof as here given.

THE PARIS REPORT ON CALCULUS IN SECONDARY SCHOOLS.

L'Enseignement for July–September contains the complete reports (referred to in the September MONTHLY) made at the Paris meeting of the International Commission on the Teaching of Mathematics. One of the two subjects there discussed, viz., the results of the introduction of the elementary notions of the calculus into secondary instruction, seems to be of such importance to the readers of the MONTHLY as to justify printing the following translation of the official summary of the general report made by Professor Beke, of Budapest, covering the replies made to the questionnaire. (For greater brevity, the differential and integral calculus will be referred to simply as the calculus.)

"1. *The place of the calculus in the secondary curriculum.*—In all countries where during the last twelve years a new plan of studies for secondary schools has been vigorously begun, a definite place of greater or less importance has been reserved for the notion of function, and as well, with very few exceptions, for the first elements of the calculus.[1]

"*A*. The elements of the infinitesimal calculus figure in the official programs of the schools, or in the plan of study established by the schools themselves, in the following countries: Bavaria, Wurtemberg, Baden, Hamburg, Austria, Denmark, France, Great Britain, Italy, Roumania, Russia, Sweden and Switzerland.

"*B*. They do not figure in the plan of study, but are elective in a large number of schools: Prussia, Saxony, Hungary, Australia; and they probably will be so before long in Holland, Norway, Belgium and Servia.

"2. *The range given to the calculus.*

"(*a*) It is nearly everywhere applied only to functions of one variable.

"(*b*) Instruction is everywhere given in the differentiation of polynomials, of rational functions (or at least quotients of two linear polynomials), and in most countries that of exponential and trigonometric functions and their inverses.

"(*c*) In most countries the notation of Lagrange is preferred to that of Leibnitz.

"(*d*) In most countries the notion of integral or primitive function is introduced. Everywhere the notion of integral follows that of derivative (in Bohemia these are treated simultaneously). In some countries the definite integral precedes the indefinite, but in most the reverse order is followed.

"3. *Applications of the infinitesimal calculus.*

"(*a*) Taylor's series figures in few programs. It is nevertheless taught in schools where the plans of study have for a long while included infinite series; in such cases the series for e^x, a^x, $\sin x$, $\cos x$, $\log(1 + x)$, $(1 + x)^m$, $\operatorname{arc\,tan} x$ are developed. Professor Beke thinks that the treatment of Taylor's series is not yet sufficiently adapted to the secondary school.

"(*b*) The calculus is applied everywhere to investigations of maxima and minima.

"(*c*) It is also applied to physics, at least for the purpose of defining velocity

[1] On the status in America see remarks by Professors Smith and Van Vleck quoted on page 326.

and acceleration, but sometimes it finds a more extended application (centers of gravity, moments of inertia, potential, etc.). In Russia only elementary mathematics is used in physics.

"(*d*) The calculus is applied in geometry to the determination of areas and volumes, and it is here that the new method renders its greatest service on the economical side. But the old methods continue to be applied, especially Cavalieri's principle.

"4. *The question of rigor.*—This is one of the most delicate points. From the side of the higher curriculum it is said that secondary teaching does more harm than good unless it adopts the rigorous method of a scientific treatment; on the other hand, the representatives of secondary instruction assert that the intellectual power of the average pupils does not permit a rigorous treatment of the calculus. Professors in secondary schools need to know the infinitesimal calculus in its modern rigorous form, but in their teaching they need to use an intuitive method, geometrical and mechanical considerations, and to progress gradually toward the necessary abstractions. This is also the surest way to arouse in the pupils the desire for rigor.

"(*a*) Irrational numbers are introduced almost everywhere incidentally in connection with the extraction of roots; the general theory is treated only in exceptional cases.

"(*b*) The notion of limit is introduced everywhere, nowhere do we rely on intuition alone. The elementary theorems relative to limits are adopted almost everywhere without explanation.

"(*c*) No allusion is made to continuous functions which nowhere admit derivatives. In certain schools instructors confine themselves to saying that at certain points the derivative may cease to exist.

"(*d*) In most schools the differential is not introduced; there is a good deal of confusion in the treatment of the notion of differential. It is greatly to be desired that the metaphysical fog of the infinitely small should not enter the secondary teaching.

"5. *Fusion of the calculus with the subject matter of the secondary school.*—The new subjects should not be given as supplementary matter alongside of the old subject matter, but a complete fusion will have to be brought about between the two. The enlargement of the rôle of the notion of function and the introduction of the calculus can be successful only if the old program be reduced and made more economical. There will then result a relief, thanks to the fusion of the new material with the old, and the suppression of old out-of-date subject matter.

"6. *The reform movement and the general opinion of educators.*—The very definite character of the results of our movement may be assured by its success and by the general opinion (l'opinion publique), always alert, of the representatives of education. The movement has gained everywhere the approval of the professors engaged in secondary education, but those engaged in higher education, who look at this from their own special point of view, do not always sympathize with our tendencies.

"We hear the complaint that a course in calculus is not followed with interest

by those who already have some knowledge of it. It is not difficult to refute this assertion. It suffices us to recall the favorable opinions which we have found among university professors in all countries, who look at our movement from a higher point of view."

To this should be added here the following extracts concerning the report of Professor D. E. Smith, the supplementary remarks made by Professor E. B. Van Vleck, and the British report made by Professor C. Godfrey.

" Professor D. E. Smith, the zealous reformer and one of the founders of the Commission, informs us, that the calculus does not figure in the secondary curriculum in the United States; it cannot even be made elective, since the pupils of the upper classes are very much absorbed in the preparation for college entrance examinations. So long as this (the calculus) is not put upon the program of these examinations, there is little chance that it will take a place in the subject matter of secondary teaching. Yet Professor Smith has the hope that before many years the calculus will be introduced in professional (technical) secondary schools. Knowing the great activity shown by our American colleagues, in the past and present, in regard to reform in mathematical teaching (we have only to recall the work of Professors D. E. Smith, Moore and Young) and seeing the immense scope of mathematical activity across the sea which dazzles our eyes and which is assuredly not without a favorable influence on the professors in secondary teaching, finally having confidence in the contact which in spite of distance exists between the workers of the two continents, we cannot doubt that before long the free development of mathematical teaching will have taken the decisive step."

Professor E. B. Van Vleck: " It may be added that to some degree the work of the first year or two of the American college course corresponds, in character, to that of the last year of the German gymnasium and the classes spéciales of the Lycées. The study of calculus is very commonly begun in the second year of the college course, and not infrequently it is taken by students in their first year. Furthermore, graphical representation for simple functions (linear and quadratic functions) has been increasingly introduced as a topic into the algebra of the high schools. From both of these facts it is clear that the tendencies now under discussion at this conference are also manifesting themselves visibly in the United States."

Professor C. Godfrey: " Broadly speaking the movement has received general support in England. Perhaps the most powerful stimulus is that of the engineers, as represented by Professor Perry. The physicists have long pressed for a modicum of calculus, and prefer to take it without too much mathematical rigor. The universities have progressively included more calculus in their examination papers for schools; these papers, together with those set by the Civil Service Commissioners (for admission to the army and the public service generally), are the most powerful lever that acts on the school curriculum. . . . Whatever opposition there has been to an introduction of the calculus at an early stage has come from those who fear that a diminished emphasis on the manipulative and formal side of algebra will have a bad effect. The question raised is this: What

algebraic equipment constitutes a firm basis for a superstructure of calculus? This is the only articulate objection that has found a voice. But the main obstacle is that most powerful force in educational matters—*vis inertiæ.*"

SUR UN PARADOXE ALGÉBRIQUE APPARENT

par G. LORIA, Université de Gênes, Italie

La contradiction signalée par M. Coolidge aux pages 184–5 de THE AMERICAN MATHEMATICAL MONTHLY de cette anneé peut se faire disparaitre de la manière suivante: Ecrivons l'equation

$$\begin{vmatrix} a & -b & c & -d \\ b & a & d & c \\ a' & -b' & c' & -d' \\ b' & a' & d' & c' \end{vmatrix} = 0$$

comme il suit

$$-\begin{vmatrix} a & -b & c & -d \\ a' & -b' & c' & -d' \\ b & a & d & c \\ b' & a' & d' & c' \end{vmatrix} = 0.$$

Appliquons a présent le théorème de Laplace sur le développement d'un déterminant et nous obtiendrons:

$$(ac' - a'c)^2 + (bd' - b'd)^2 + (ad' - a'd)^2 + (bc' - b'c)^2 - 2(a'b - ab')(c'd - cd') = 0.$$

ou bien, par des transformations algébriques faciles,

$$[(ac' - a'c) - (bd' - b'd)]^2 + [(ad' + bc') - (a'd + b'c)]^2 = 0.$$

Comme a, b, . . . sont toutes des quantités réelles cette équation unique se décompose dans les deux suivantes,

$$(ac' - a'c) = (bd' - b'd),$$

$$(ad' + bc') = (a'd + b'c),$$

qui sont précisément celles qu'a trouvees l'éminent professeur de l'Harvard University.

GÊNES, 28, VII, 1914.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

Dialogues Concerning Two New Sciences. By GALILEO GALILEI. Translated from the Italian and Latin into English by HENRY CREW and ALFONSO DE SALVIO with an introduction by ANTONIO FAVARO. The Macmillan Company, New York, 1914. xxi+300 pages. $2.00.

Into these dialogues, Galileo gathered, during his last years, practically all of his work that is of value to the physicist or the engineer. The translation into English, at this time, is most desirable as the few copies of the two former translations, by Thomas Salusbury in 1665 and Thomas Weston in 1730, are quite inaccessible and many passages are either ambiguous or unintelligible to the modern reader.

The translation strictly follows the text of the National Edition of the Works of Galileo recently completed by Antonio Favaro. The text is essentially the same as that of the original Elzevir edition of 1638. All comments and annotations have been omitted with the exception of a few footnotes properly labeled (Trans.) in order to preserve the original as nearly as possible, and the translation is as literal as is consistent with clearness and modernity. Where there is any deviation from this rule and in case of some technical terms the original Italian or Latin phrase is added enclosed in brackets. The original illustrations are reproduced.

The presentation of the science in the form of dialogues is exceedingly striking and effective. The dialogues continue during four successive days. In the first day the three men discuss resistance which solid bodies offer to fracture, friction, cohesion, the finite and the infinite, laws of falling bodies, terminal velocity, the density of air, isochronous vibrations of the pendulum, resonance, etc. During the second day the same men discuss levers, beams, cohesion, resistance to fracture, and strength of hollow tubes.

On the third day they begin the discussion of the "second new science," treating of motion. They consider the laws of uniform motion and of uniformly accelerated motion, and the time of descent of a body down an inclined plane under various conditions.

The fourth day is taken up with the consideration of the composition of a horizontal uniform motion with a vertical uniformly accelerated motion, the velocity of a projectile at any point of its path, the maximum range and various other problems concerning projectiles, the sagging of a stretched cord. The term momentum is used and impulsive forces are mentioned but not explained.

It will repay anyone, who has not already done so, to read this admirable book of historic as well as intrinsic interest.

ANTHONY ZELENY.

UNIVERSITY OF MINNESOTA.

Plane Trigonometry and Applications. By E. J. WILCZYNSKI, Professor of Mathematics in the University of Chicago. Edited by PROFESSOR H. E. SLAUGHT. Allyn and Bacon, 1914. xi+265 pages. $1.25.

Logarithmic and Trigonometric Tables. By the same author and editor. Allyn and Bacon, 1914. xx+97 pages. $1.00.

Every friend of mathematics and every friend of mathematical teaching must rejoice when scholars of national and international reputation give their time to the compiling of text-books in the more elementary branches of the science. It is a serious mistake to suppose that high school teachers are best equipped to write high school texts. It is particularly fortunate when the author is not only a scholar of wide outlook, but also a man with a keen instinct for teaching. When, moreover, the work of such a man has passed under the searching eye of an editor who is also an acknowledged leader in mathematical teaching, it is natural to expect much in advance of the book.

The work is divided into two parts, the first of which is devoted to the solution of triangles. The trigonometric functions of acute angles are defined as usual, and the notion of the general angle is deferred till part two. This is precisely the method which a less experienced teacher would avoid. Why not give the general definition at the very first, and avoid the readjusting of ideas made necessary by two different definitions? This is, indeed, a very vital and important matter in pedagogy. There is a tendency of late years to hurry the child into algebra before it has had time to get acquainted with the simpler conceptions of arithmetic. Why study arithmetic when you might as well be studying the more general science? Why not, indeed? And why not study the theory of groups in the grades, seeing that algebra is, in the last analysis, a chapter in abstract groups? The reviewer looks with admiration on the skilful means by which all of the formulæ for the solution of triangles, both right and oblique, are developed without the assistance of coördinate geometry, and even without the use of the addition theorems. Even the law of tangents has been derived by purely synthetic methods. Professor Wilczynski has hit upon the happy device of giving a central position to the area problem. All the formulæ of trigonometry present themselves naturally with a unifying point of attack. This part is also published separately and would seem to be admirably adapted for use in secondary schools.

Much attention is given in Part One to the subject of computation. The author is a computor of many years experience and can speak with authority on this subject. The *negligence* of expressing a result in fewer decimal places than one is able to guarantee, and the *dishonesty* of expressing it in more are topics dwelt on with some warmth. The reviewer is a little disconcerted, however, to find the so-called *abbreviated method* of multiplication referred to as preferable to the ordinary method. They are both so very bad that it is difficult to say which is worse.

There is also in this chapter a lucid discussion of logarithms, which with the illustrative examples will be easily grasped by any class. A short description is also given of the slide-rule.

The book of tables published with the trigonometry should be noted in connection with part one. It is a very attractively constructed table of five-place logarithms, both of numbers and of trigonometric functions. The spacing is generous and well calculated to relieve the eye of undue strain. A number of smaller tables are also given in the book, such as extra tables for " small " angles; a table of four places for the natural functions; a table of squares; a table for transforming angles; a list of constants; and three small three-place logarithmic tables. Altogether a very useful book of tables.

Part Two of the trigonometry contains, besides the discussion of the general angle, and the addition theorems and consequences of them, certain chapters not usually met with in treatises on the elements of this subject. The chapter on directed line-segments is a good introduction for the study of projective geometry and for the study of vector analysis. The chapter on inverse functions is one that will be of value to every student who will go on to the study of the calculus. It is a refreshing thing to see an American text-book with the notation " Arc sine."

Perhaps the most unusual feature of the book is the insertion of the chapter on the Theory of Wave Motion which closes the book. It is a subject of unusual interest and importance, and the chapter is one of the most beautiful in the book. We doubt, however, whether many teachers will venture to include it in a course where students are apt to have as little mathematical maturity as most students of trigonometry do. Such things, however, are worth while placing where teachers and students can have a look at them sometimes; and who knows whether twenty years from now they may be included in the earlier years of the student's life just as now we include the study of the calculus in the freshman and sophomore years and not in the very last year as was the custom twenty years ago.

Much of interest in historical matter has been introduced " not in the form of detached historical notes, but organically connected with the topic under discussion." This appears to the reviewer to be a decided improvement on the usual method, but he cherishes the belief that the effect would be much better if the historical matter were collected into a separate chapter at the end of the book, and the subject discussed as a whole,—but each one to his taste in such matters.

The reviewer has not given the book a microscopic examination for printer's errors, and mis-spelled words. He has not even worked out any of the trigonometric identities, of which he is glad to note the fewness, but he has been impressed with the inviting appearance of the pages, and the successful arrangement of the matter on the page. These details count for much more in making a usable book than teachers sometimes think.

D. N. Lehmer.

University of California.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

PROBLEMS FOR SOLUTION.

ALGEBRA.

When this issue was made up no solution of 420 had been received.

424. Proposed by S. A. JOFFE, New York City.

Sum the series

$$\binom{n}{a}-\binom{n-1}{a}\binom{i}{1}+\binom{n-2}{a}\binom{i}{2}-\binom{n-3}{a}\binom{i}{3}+\cdots+(-1)^i\binom{n-i}{a},$$

and consider the cases $i = a$ and $i > a$.

425. Proposed by CLIFFORD N. MILLS, Brookings, S. D.

Solve for x and y the equations: $2^{x+y} = 6$ and $2^{x+1} = 3^y$.

GEOMETRY.

When this issue was made up solutions had been received for numbers 370, 430, 432–3, 447–8, 450–1. Please give attention to 427, 442, 446, 449.

452. Proposed by NATHAN ALTSHILLER, University of Washington.

Through a given point a secant is drawn that meets three given concurrent lines in the points A, B, C respectively. Determine the position of the secant by the condition $AB/BC = K$, K being given.

453. Proposed by CLIFFORD N. MILLS, South Dakota Agricultural College.

Prove geometrically the formulæ for $\sin 2\beta$, $\cos 2\beta$, $\sin 3\beta$, $\cos 3\beta$.

454. Proposed by LOUIS ROUILLION, Mechanics Institute, New York City.

Show how to construct an equilateral triangle with its vertices lying on three lines not equally spaced.

CALCULUS.

When this issue was made up solutions had been received for numbers 358–9, 361–2, 364, 371, 373. Please give attention to 332, 339, 340, 342, 348, 353, 360, 363.

374. Proposed by C. N. SCHMALL, New York City.

Show that, on a *Mercator's Chart*, a great circle of a sphere whose radius is r will be represented by a curve whose equation is of the form

$$c(e^{y/r} - e^{-(y/r)}) = 2 \sin\left(\frac{x}{r} + \theta\right).$$

(*Note.* See EISENHART'S *Differential Geometry*, § 46, pp. 107–108; OSGOOD'S *Calculus*, pp. 331–333, § 6.)

375. Proposed by V. M. SPUNAR, Chicago, Illinois.

Solve the differential equation,

$$x^2(a-bx)\frac{d^2y}{dx^2} - 2x(2a-bx)\frac{dy}{dx} + 2(3a-bx)y = 6a^2.$$

MECHANICS.

When this issue was made up solutions had been received for numbers 292, 293, 295–6–7–8–9. Please give attention to 272, 277–8–9, 286–7, 290–1, 294, 300.

301. Proposed by CLIFFORD N. MILLS, Brookings, S. D.

A wire is hanging from two points in the same horizontal plane. If the difference between the length of the wire and the actual distance between the supports is very small, show that

$$s = x\left(1 + \frac{x^2}{6c^2}\right),$$

where s is one half the length of the wire, c is the horizontal tension at the lowest point divided by w the load per unit of horizontal distance, and x is the distance of lowest point of the curve to the point of support.

NUMBER THEORY.

When this issue was made up solutions had been received for numbers 212–13, 215–16, 218, 220, 223. Please give attention to 191–2, 196, 198, 201–2, 205, 208–9–10–11, 214, 217, 219, 221–2.

224. Proposed by PATRICK WALSH, New Orleans, Louisiana.

Find the sides, in rational numbers, of a right angled triangle whose area is $5\frac{1}{2}$.

225. Proposed by W. DE W. CAIRNS, Oberlin College.

L'Intermédiaire for June, 1914, contains the following problem:

"If we write the terms of the arithmetic series 1, 5, 9, 13, 17, 21, 25, 29, 33, $\cdots$ as follows:

$$\begin{array}{ccccccc} 1 \\ 5 & 9 & 13 \\ 17 & 21 & 25 & 29 & 33 \\ 37 & 41 & 45 & 49 & 53 & 57 & 61 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot, \end{array}$$

it is seen that the sum of the terms of each line is a cube, and that these are the cubes of the successive "odd integers. How is this shown?"

It is here proposed not only to prove this, but to generalize the theorem as suggested, using, however, the simpler (and better known) case which includes all of the successive integers:

$$\begin{array}{cccc} 1 \\ 3 & 5 \\ 7 & 9 & 11 \\ 13 & 15 & 17 & 19 \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

SOLUTIONS OF PROBLEMS.

ALGEBRA.

412. Proposed by H. L. SLOBIN, University of Minnesota.

Form the algebraic equation whose roots are:

$$a_1 = \cos\frac{\pi}{9}, \qquad a_2 = -\cos\frac{2\pi}{9}, \qquad a_3 = -\cos\frac{4\pi}{9}.$$

SOLUTION BY CLIFFORD N. MILLS, Brookings, South Dakota.

From the suggestions given by the proposer,

$$\cos r\pi = \frac{(-1)^r + (-1)^{-r}}{2}.$$

Hence,

$$a_1 = \frac{(-1)^{1/9}+(-1)^{-1/9}}{2},\quad a_2 = -\frac{(-1)^{2/9}+(-1)^{-2/9}}{2},\quad a_3 = -\frac{(-1)^{4/9}+(-1)^{-4/9}}{2},$$

and the equation to be constructed is

$$x^3 - (a_1 + a_2 + a_3)x^2 + (a_1a_2 + a_1a_3 + a_2a_3)x - a_1a_2a_3 = 0.$$

We have

$$\Sigma a_1 = \frac{-(-1)^{8/9} - (-1)^{6/9} + (-1)^{5/9} + (-1)^{3/9} - (-1)^{2/9} - 1}{2(-1)^{4/9}},$$

which involves the nine 9th roots of (-1).

From $x^9 + 1 = 0$ we have

$$x^8 - x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = 0,$$

which contains all the *complex* 9th roots of (-1).

Hence

$$x^8 + x^6 - x^5 - x^3 + x^2 + 1 = x^7 - x^4 + x,$$

and this is equal to zero since

$$x^7 - x^4 + x = x(x^6 - x^3 + 1),$$

in which the factor $x^6 - x^3 + 1 = 0$.

Therefore

$$\Sigma a_1 = \frac{-(-1)^{1/9}[(-1)^{6/9} - (-1)^{3/9} + 1]}{2(-1)^{4/9}} = 0.$$

Again,

$$a_1a_2 + a_1a_3 + a_2a_3 = -\frac{1}{4}\left[\frac{(-1)^{6/9} + (-1)^{4/9} + (-1)^{2/9} + 1}{(-1)^{3/9}}\right.$$
$$\left. + \frac{(-1)^{10/9} + (-1)^{8/9} + (-1)^{2/9} + 1}{(-1)^{5/9}} - \frac{(-1)^{12/9} + (-1)^{8/9} + (-1)^{4/9} + 1}{(-1)^{6/9}}\right].$$

Hence

$$\Sigma a_1a_2 = \frac{1}{4}\,\frac{\begin{array}{l}(-1)^{12/9} - (-1)^{11/9} - 2(-1)^{9/9} + (-1)^{8/9} - (-1)^{7/9} \\ \qquad - (-1)^{5/9} + (-1)^{4/9} - 2(-1)^{3/9} - (-1)^{1/9} + 1\end{array}}{(-1)^{6/9}}.$$

Now

$$x^8 - x^7 - x^5 + x^4 - x^3 - x + 1 = -x^6 - x^2.$$

Hence

$$\Sigma a_1a_2 = \frac{1}{4}\,\frac{x^9 - 2x^6 - x^3 + 1}{x^3} = \frac{1}{4}\,\frac{x^9 - 3x^3 + 1}{x^3} = -\frac{3}{4},$$

making use of $x^9 + 1 = 0$ and $x^6 = x^3 - 1$.

Finally,

$$a_1a_2a_3 = \frac{(-1)^{14/9} + (-1)^{13/9} + (-1)^{10/9} + (-1)^{8/9} + (-1)^{6/9} + (-1)^{4/9} + (-1)^{2/9} + 1}{8(-1)^{7/9}}$$
$$= \frac{-(-1)^{5/9} - (-1)^{3/9} - (-1)^{1/9} + (-1)^{7/9} + (-1)^{5/9} + (-1)^{3/9} + (-1)^{1/9}}{8(-1)^{7/9}}$$
$$= \frac{(-1)^{7/9}}{8(-1)^{7/9}} = \frac{1}{8},$$

making use of $x^8 + x^6 + x^4 + x^2 + 1 = x^7 + x^5 + x^3 + x$.

Hence, the required equation is

$$x^3 - \tfrac{3}{4}x - \tfrac{1}{8} = 0.$$

Solved similarly by H. C. FEEMSTER, and also by C. E. GITHENS by merely substituting the values of a_1, a_2, a_3 and reducing by means of trigonometric relations. Mr. Feemster also sent in such a second solution.

Editorial note. The details of the work on this problem may be simplified by use of Gauss' periods. See WEBER's *Algebra*, volume I, chapter XVI.

413. Proposed by C. N. SCHMALL, New York City.

Apply Euler's transformation to show that

$$1 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots = \frac{1+x}{(1-x)^3}.$$

SOLUTION BY W. C. BRENKE, University of Nebraska.

This is a special case of the series

$$S(x) = 1 + 2^rx + 3^rx^2 + 4^rx^3 + \cdots; \; r = \text{positive integer}.$$

Such series may be summed by repeated use of Euler's transformation based on the factor $(1 - x)$. The whole operation may be performed at once as follows, the process being valid for $|x| < 1$.

Multiply both sides of the last equation by

$$(1 - x)^{r+1} = 1 - {}_{r+1}C_1x + {}_{r+2}C_2x^2 - \cdots + (-1)^{r+1}x^{r+1}.$$

Arranging the right member in powers of x, we find that the coefficients of x^{r+1}, x^{r+2}, $\cdots$ are all zero by virtue of the formula (special case of (3) p. 183 of Chrystal's Algebra, Vol. 2)

$$n^r - (n-1)^r{}_nC_1 + (n-2)^r{}_nC_2 - \cdots + (-1)^{n-1}{}_nC_{n-1} = 0; \; r < n.$$

Hence

$$(1 - x)^{r+1}S(x) = 1 + (2^r - {}_{r+1}C_1)x + (3^r - 2^r{}_{r+1}C_1 + {}_{r+1}C_2)x^2 + \cdots$$
$$+ (r^r - (r-1)^r{}_{r+1}C_1 + \cdots + (-1)^r{}_{r+1}C_r)x^r.$$

The result may also be written in the form

$$S(x) = \frac{S_r(x) - {}_{r+1}C_1xS_{r-1}(x) + {}_{r+1}C_2x^2S_{r-2}(x) + \cdots + (-1)^r{}_{r+1}C_rx^r}{(1-x)^{r+1}},$$

where

$$S_n(x) = 1 + 2^rx + 3^rx^2 + \cdots + (n+1)^rx^n.$$

When $r = 2$ this gives the sum of the proposed series.

Also solved by the PROPOSER.

414. Proposed by R. D. CARMICHAEL, Indiana University.

Prove by means of an example that one of the series

$$\sum_{k=1}^{\infty} \frac{1}{c_k}, \quad \sum_{k=1}^{\infty} \frac{1}{c_k - 1}, \qquad c_k \neq 0, 1,$$

may be divergent while the other is convergent.

SOLUTION BY E. B. WILSON, Massachusetts Institute of Technology.

The following two series fit the conditions of the problem:

$$\frac{1}{\sqrt{2}}+\frac{1}{-\sqrt{3}}+\frac{1}{\sqrt{4}}+\frac{1}{-\sqrt{5}}+\cdots,$$

$$\frac{1}{\sqrt{2}-1}+\frac{1}{-\sqrt{3}-1}+\frac{1}{\sqrt{4}-1}+\frac{1}{-\sqrt{5}-1}+\cdots.$$

The first converges because it is an alternating series whose terms steadily decrease in absolute value and approach zero. That the second is divergent may be seen by bracketing the terms in pairs. For, if n is even, we have

$$\frac{1}{\sqrt{n}-1}+\frac{1}{-\sqrt{n+1}-1}=\frac{-\sqrt{n}+\sqrt{n+1}+2}{(\sqrt{n}-1)(\sqrt{n+1}-1)}>\frac{2}{n+1};$$

whence the series is divergent as one sees by comparison with the divergent series

$$\frac{2}{3}+\frac{2}{5}+\frac{2}{7}+\cdots.$$

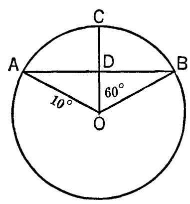
GEOMETRY.

440. Proposed by W. L. WATSON, Moundsville, West Va.

A solid sector is cut out of a sphere 10 feet in radius by a cone whose vertical angle is 120°. Find the radius of the sphere whose volume is equal to that of the sector.

SOLUTION BY EMMA M. GIBSON, Drury College.

The volume of the solid sector O–ABC is equal to that of the cone AOB, having a vertex angle of 120° and altitude OD equal to 10 cos 60°, plus the volume of a spherical segment of altitude $CD = 10 - 10\cos 60° = 5$, and diameter of base $AB = 2 \times 10 \sin 60° = 10\sqrt{3}$. Then



$$\text{Volume of cone} = \frac{1}{3}\text{base} \times \text{altitude} = \frac{1}{3}\pi 75 \times 5 = 125\pi.$$

$$\text{Volume of segment} = \frac{1}{3}\pi \times 25 \times 25 = \frac{625}{3}\pi.$$

$$\therefore \text{Volume of solid sector} = 125\pi + \frac{625\pi}{3} = \frac{1000\pi}{3}.$$

Let r' be the radius of sphere whose volume is equal to the volume of the solid sector. Then $\frac{4}{3}\pi r'^3 = \frac{1000}{3}\pi$.

$$\therefore\ r' = \sqrt[3]{250} = 5\sqrt[3]{2}.$$

Also solved by A. M. HARDING, PAUL CAPRON, WALTER C. EELLS, ELMER SCHUYLER, H. C. FEEMSTER, HORACE OLSON, C. E. GITHENS, GEO. W. HARTWELL, and CLIFFORD N. MILLS.

CALCULUS.

354. Proposed by C. N. SCHMALL, New York City.

Prove

$$\Gamma(1+a)\Gamma(1-a) = \frac{\pi a}{\sin \pi a}.$$

SOLUTION BY A. G. CARIS, Defiance, O.

(The outline for this proof is taken from notes of a course in *Definite Integrals* given by Professor G. A. Bliss at the University of Chicago.)

The following preliminary theorems are used in the proof.

(1) $$\sin z = z\prod_{\nu=1}^{\infty}\left(1 - \frac{z^2}{\nu^2\pi^2}\right).$$

(2) $$\cos z = \prod_{\nu=1}^{\infty}\left(1 - \frac{4z^2}{(2\nu-1)^2\pi^2}\right).$$

(3) $$\frac{\pi}{\sin \pi a} = \frac{1}{a} + \frac{2a}{1^2 - a^2} - \frac{2a}{2^2 - a^2} + \frac{2a}{3^2 - a^2} - \cdots.$$

(4) $$\int_0^{\alpha} \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin \pi a}.$$

(5) $$\Gamma(1+a) = a\Gamma(a).$$

(6) $$y^{-a}\Gamma(a) = \int_0^{\infty} x^{a-1}e^{-yx}dx.$$

PROOF OF THEOREM (1).

$$\sin z = 2\sin\frac{z}{2}\cos\frac{z}{2} = 2\sin\frac{z}{2}\sin\frac{z+\pi}{2}.$$

By repeated applications of this formula we obtain

$$\sin z = 2^{2^n-1}\sin\frac{z}{2^n}\cdot\sin\frac{z+\pi}{2^n}\cdot\sin\frac{z+2\pi}{2^n}\cdots\sin\frac{z+(2^n-1)\pi}{2^n}.$$

This may be written more briefly by substituting $2^n = p$.

$$\sin z = 2^{p-1}\prod_{\nu=0}^{p-1}\sin\frac{z+\nu\pi}{p}.$$

Combine these factors in pairs, taking together such factors that the sum of the coefficients of π shall be p. The general form of these products will be

$$\sin\frac{z+\nu\pi}{p}\sin\frac{z+(p-\nu)\pi}{p}.$$

By trigonometric transformation, this becomes

$$\sin^2\frac{\nu\pi}{p}-\sin^2\frac{z}{p}.$$

There will be $\left(\frac{p}{2}-1\right)$ of these factors, the factor $\sin\frac{z}{p}$, and the factor $\sin\dfrac{z+\frac{p}{2}\pi}{p}$ which equals $\cos\frac{z}{p}$.

Whence,
$$\sin z = 2^{p-1}\sin\frac{z}{p}\cos\frac{z}{p}\prod_{\nu=1}^{(p/2)-1}\left(\sin^2\frac{\nu\pi}{p}-\sin^2\frac{z}{p}\right). \tag{A}$$

Dividing both sides of this equation by z,

$$\frac{\sin z}{z}=\frac{2^{p-1}}{p}\cdot\frac{\sin\frac{z}{p}}{\frac{z}{p}}\cos\frac{z}{p}\prod_{\nu=1}^{(p/2)-1}\left(\sin^2\frac{\nu\pi}{p}-\sin^2\frac{z}{p}\right).$$

Passing to the limit of both sides as $z \doteq 0$, we have

$$1=\frac{2^{p-1}}{p}\prod_{\nu=1}^{(p/2)-1}\sin^2\frac{\nu\pi}{p}.$$

Dividing both members of (A) by this result, we have

$$\sin z = p\sin\frac{z}{p}\cos\frac{z}{p}\prod_{\nu=1}^{(p/2)-1}\left(1-\frac{\sin^2\frac{z}{p}}{\sin^2\frac{\nu\pi}{p}}\right).$$

Taking the limit of this expression as $p \doteq \infty$, we have

$$\sin z = z\prod_{\nu=1}^{\infty}\left(1-\frac{z^2}{\nu^2\pi^2}\right).$$

Proof of Theorem (2).

$$\sin z = 2\sin\frac{z}{2}\cos\frac{z}{2}=z\prod_{\nu=1}^{\infty}\left(1-\frac{z^2}{\nu^2\cdot\pi^2}\right).$$

From (1),

$$\sin\frac{z}{2}=\frac{z}{2}\prod_{\nu=1}^{\infty}\left(1-\frac{z^2}{(2\nu)^2\pi^2)}\right).$$

Dividing the first of these two equations by the second, we have

$$2\cos\frac{z}{2} = 2\prod_{\nu=1}^{\infty}\left(1 - \frac{z^2}{(2\nu-1)^2\pi^2}\right).$$

From this we have at once

$$\cos z = \prod_{\nu=1}^{\infty}\left(1 - \frac{4z^2}{(2\nu-1)^2\pi^2}\right).$$

PROOF OF THEOREM (3).

From (1) and (2) we have

$$\sin \pi a = \pi a \prod_{\nu=1}^{\infty}\left(1 - \frac{a^2}{\nu^2}\right), \qquad \cos \pi a = \prod_{\nu=1}^{\infty}\left(1 - \frac{4a^2}{(2\nu-1)^2}\right).$$

At once we have

$$\log \sin \pi a = \log \pi + \log a + \log\left(1 - \frac{a^2}{1^2}\right) + \log\left(1 - \frac{a^2}{2^2}\right) + \cdots$$

and

$$\log \cos \pi a = \log\left(1 - \frac{4a^2}{1^2}\right) + \log\left(1 - \frac{4a^2}{3^2}\right) + \log\left(1 - \frac{4a^2}{5^2}\right) + \cdots.$$

By differentiation of these two series, we have

$$\pi \cot \pi a = \frac{1}{a} - \frac{2a}{1^2 - a^2} - \frac{2a}{2^2 - a^2} - \frac{2a}{3^2 - a^2} - \cdots$$

$$\pi \tan \pi a = \frac{8a}{1^2 - 4a^2} + \frac{8a}{3^2 - 4a^2} + \frac{8a}{5^2 - 4a^2} + \cdots.$$

Substituting $a/2$ for a, we have

$$\pi \tan \frac{\pi a}{2} = \frac{4a}{1^2 - a^2} + \frac{4a}{3^2 - a^2} + \frac{4a}{5^2 - a^2} + \cdots.$$

$$\pi \tan \frac{\pi a}{2} + \pi \cot \pi a = \pi\left[\frac{\sin\frac{\pi a}{2}}{\cos\frac{\pi a}{2}} + \frac{\cos^2\frac{\pi a}{2} - \sin^2\frac{\pi a}{2}}{2\sin\frac{\pi a}{2}\cos\frac{\pi a}{2}}\right] = \frac{\pi}{\sin \pi a}.$$

Adding the corresponding series, we have

$$\frac{\pi}{\sin \pi a} = \frac{1}{a} + \frac{2a}{1^2 - a^2} - \frac{2a}{2^2 - a^2} + \frac{2a}{3^2 - a^2} - \cdots.$$

PROOF OF THEOREM (4).

$$\int_0^\alpha \frac{x^{a-1}}{1+x}\,dx = \int_0^1 \frac{x^{a-1}}{1+x}\,dx + \int_1^\alpha \frac{x^{a-1}}{1+x}\,dx.$$

Transforming the second integral by the substitution, $x = \frac{1}{y}$,

$$\int_1^a \frac{x^{a-1}}{1+x}\,dx = \int_0^1 \frac{y^{-a}}{1+y}\,dy,$$

$$\int_0^1 \frac{x^{a-1}}{1+x}\,dx = \int_0^1 \{x^{a-1} - x^a + x^{a+1} - x^{a+2} + \cdots\}\,dx,$$

$$\int_0^1 \frac{y^{-a}}{1+y}\,dy = \int_0^1 \{y^{-a} - y^{-a+1} + y^{-a+2} - \cdots\}\,dy.$$

Integrating these two series term by term, we have,

$$\int_0^a \frac{x^{a-1}}{1+x}\,dx = \left\{\frac{1}{a} - \frac{1}{a+1} + \frac{1}{a+2} - \cdots\right\} + \left\{\frac{1}{1-a} - \frac{1}{2-a} + \frac{1}{3-a} - \cdots\right\}$$

$$= \frac{1}{a} + \frac{2a}{1^2 - a^2} - \frac{2a}{2^2 - a^2} + \frac{2a}{3^2 - a^2} - \cdots.$$

But this, by (3), is $\frac{\pi}{\sin \pi a}$.

$$\therefore \quad \int_0^a \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin \pi a}.$$

Proofs of theorems (5) and (6) may be found in almost any elementary discussion of the properties of $\Gamma(a)$, and need not be given here.

We are now to prove

$$\Gamma(1+a)\Gamma(1-a) = \frac{\pi a}{\sin \pi a}.$$

By (5), $\Gamma(1+a) = a\Gamma(a)$.

Our problem is thus reduced to the proof of

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a}.$$

By (6),
$$\frac{\Gamma(a)}{y^a} = \int_0^a x^{a-1}e^{-yx}\,dx.$$

Multiplying both sides by e^{-y}, we have

$$\frac{e^{-y}\Gamma(a)}{y^a} = \int_0^a x^{a-1}e^{-y(x+1)}\,dx.$$

Taking the integral of both sides with respect to y from 0 to a, we have

$$\Gamma(a)\int_0^a \frac{e^{-y}}{y^a}\,dy = \int_0^a\int_0^a x^{a-1}e^{-y(x+1)}\,dx\,dy = \int_0^a x^{a-1}\int_0^a e^{-y(x+1)}\,dy\,dx$$

$$= \int_0^a \frac{x^{a-1}}{1+x}\,dx.$$

By (4),

$$\int_0^\infty \frac{x^{a-1}}{1+x}dx = \frac{\pi}{\sin \pi a}.$$

$$\int_0^\infty \frac{e^{-y}}{y^a}dy = \int_0^\infty y^{-a}e^{-y}dy = \int_0^\infty y^{(1-a)-1}e^{-y}dy = \Gamma(1-a).$$

$$\therefore \quad \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a},$$

and

$$\Gamma(1+a)\Gamma(1-a) = \frac{\pi a}{\sin \pi a}.$$

Also solved by T. M. Blakslee, C. N. Schmall, A. M. Harding, A. L. McCarty, and J. W. Clawson.

355. Proposed by C. N. SCHMALL, New York City.

Given the curve of the nth degree,

$$y^n - (a + bx)y^{n-1} + (c + dx + ex^2)y^{n-2} + \cdots = 0,$$

show that if each ordinate is divided by the corresponding subtangent, the sum of all the resulting ratios will be a constant.

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

The question should read: "show that if for a given abscissa each ordinate . . . ."

The sum of all the ordinates corresponding to a given abscissa x_1 is equal to minus the coefficient of y^{n-1}, viz., $+ (a + bx_1)$.

Hence, the sum of the derivatives of the several ordinates will be $+ b$.

But each subtangent is the ordinate divided by the slope of the curve at the top of the ordinate. Hence any ordinate divided by its corresponding subtangent is equal to the slope of the curve at the top of that ordinate. We have just shown that the sum of these slopes for all the ordinates corresponding to a given abscissa is a constant. This proves the problem, as amended. See Edwards' *Differential Calculus,* page 151.

Also solved by the Proposer.

MECHANICS.

274. Proposed by G. B. M. ZERR.

A sphere moves on the concave side of a rough cylindrical surface of which the transverse section perpendicular to the generating lines is a hypocycloid. If $s = l \sin n\theta$ be the intrinsic equation of the hypocycloid, then $l = (a - b)4b/a$, $n = a/(a - 2b)$, where a = radius of fixed circle, b = radius of rolling circle.

REMARK BY A. H. WILSON, Haverford College.

The statement of this problem is incomplete. As it stands it is not a problem at all.

275. Proposed by W. J. GREENSTREET, Editor of the Mathematical Gazette, England.

If a particle be attracted towards the angular points of a regular hexagon by forces equal to r^{-h}, at distance r, find the condition for stability of equilibrium.

SOLUTION BY A. H. WILSON, Haverford College.

The discussion is for the center of the hexagon, an obvious position of equilibrium. Let this point be taken as origin, the hexagon placed with two vertices on the x-axis. Let r_0 be the side of the hexagon.

Form the potential function V. For one attracting center this is of the form $\int_{r_0}^{r} r^{-h}dV = r^{1-h}/(1-h) - r_0^{1-h}/(1-h)$. (The value $h = 1$ must be excepted.) For simplicity the origin is taken as a standard position.

$$V = \Sigma r^{1-h}/(1-h) - V_0.$$

Each r is of the form $[(x-s)^2 + (y-t)^2]^{1/2}$. Expand V in the neighborhood of the origin in powers of x and y. For r^{1-h} this expansion is

$$r^{1-h} = r_0^{1-h} + s(1-h)r_0^{-h-1}x + t(1-h)r_0^{-h-1}y + \tfrac{1}{2}[(1-h)r_0^{-h-1} - (1-h^2)s^2r_0^{-h-3}]x^2$$
$$-(1-h^2)r_0^{-h-3}st\cdot xy + \tfrac{1}{2}[(1-h)r_0^{-h-1} - (1-h^2)r_0^{-h-3}t^2]y^2 + \cdots.$$

For (s, t) substitute successively the coördinates of the hexagon points and sum:

$$V = V_0 - \tfrac{1}{2}(2+3h)r_0^{-h-1}(x^2+y^2) + \cdots.$$

In the neighborhood of the origin then the equations of motion are

$$m\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x} = (2+3h)r_0^{-h-1}x,$$

$$m\frac{d^2y}{dt^2} = -\frac{\partial V}{\partial y} = (2+3h)r_0^{-h-1}y.$$

For stable equilibrium we must have here

$$2 + 3h < 0.$$

MISCELLANEOUS QUESTIONS.

EDITED BY R. D. CARMICHAEL.

NEW QUESTIONS.

19. How many known proofs are there of the proposition that the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides? Where are the proofs to be found?

20. Some of our readers would like to have a simple account, without proofs, of just what has been accomplished toward the proof of the theorem that the equation $x^n + y^n = z^n$ is impossible in integers when $n > 2$.

21. For the diophantine equation

$$x^2 - y^3 = 17$$

$$u_2 = \frac{a^2x - x^3}{c}(e^{cx} + e^{-cx})\Big|_0^a - \frac{1}{c}\int_0^a (a^2 - 3x^2)(e^{cx} + e^{-cx})dx$$

$$= -\frac{1}{c^2}\left[(a^2 - 3x^2)(e^{cx} - e^{-cx})\Big|_0^a + 6\int_0^a x(e^{cx} - e^{-cx})dx\right]$$

$$= -\frac{1}{c^2}\left[-2a^2(e^{ac} - e^{-ac}) + \frac{6}{c}\left\{a(e^{ac} + e^{-ac}) - \frac{1}{c}(e^{ac} - e^{-ac})\right\}\right]$$

$$= \frac{1}{c^2}\left[2a^2(e^{ac} - e^{-ac}) - \frac{6a}{c}(e^{ac} + e^{-ac}) + \frac{6}{c^2}(e^{ac} - e^{-ac})\right].$$

Substituting the values of u_1 and u_2 in the expression above for u we obtain the desired result.

NOTES AND NEWS.

EDITED BY W. DEW. CAIRNS.

"The propagation of electric waves in wireless telegraphy" is the title of a paper by Professor George R. Dean, of the Missouri School of Mines, in *The Electrician* for September 11, 1914.

Dr. L. L. DINES, formerly professor of mathematics at the University of Arizona, has been elected to an associate professorship at the University of Saskatchewan.

The Southwestern Section of the American Mathematical Society held its eighth regular meeting at the University of Nebraska on Saturday, November 28, 1914.

Mr. J. L. RILEY, formerly instructor in mathematics at the University of Oklahoma, has accepted a fellowship in Rice Institute, Houston, Texas, for the year 1914–15. Professor E. P. R. DUVAL has returned to his former position as associate professor of mathematics at the University of Oklahoma.

Dr. E. E. Whitford has been promoted to an assistant professorship of mathematics at the College of the City of New York.

Mr. Joel D. Eshleman, A.B., has been appointed instructor in mathematics in Adelbert College, Western Reserve University.

Dr. E. B. Stouffer, formerly instructor in mathematics in the University of Illinois, has accepted an assistant professorship in the University of Kansas.

Professor M. Frechet, University of Poitiers, France, who was to lecture at the University of Illinois during the present year, has joined the French army.

The Cambridge University Press of England has published "Tables for Statisticians and Biometricians" by Karl Pearson in quarto form, 228 pages.

The University of Chicago Press announces as in press "The Theory of Collineation Groups" by Professor H. F. Blichfeldt, of Leland Stanford Junior University.

A second revised edition of the English translation by Dr. G. G. Morrice of Professor Felix Klein's well-known "Lectures on the Icosahedron and the Solution of Equations of the Fifth Degree" has just been issued from the press of Kegan Paul, Paris; Trubner and Co., London.

The majority of the American reports to the International Commission on the Teaching of Mathematics are no longer given free distribution by the U. S. Bureau of Education; it is therefore necessary to correct the statement made at the end of the September MONTHLY. On application, however, to the Commissioner of Education, Washington, D. C., a bulletin of the publications of the Bureau will be sent free, giving the prices at which these may be obtained. The whole set of reports above mentioned costs less than one dollar.

At the University of Chicago the departments of mathematics and mathematical astronomy combine as one body, on the one hand in the biweekly club meeting for reports of research in progress, and on the other hand in certain stated meetings for social intercourse. Under the latter head, a general dinner is held in the autumn for the advanced students and members of the staff, a dinner in the spring for the fellows of the departments provided by the members of the staff, and a general dinner in the summer for advanced students and staff. The members of the staff also meet regularly each month for a dinner and evening discussion of matters pertaining to the departments. The autumn general dinner was held on Wednesday evening, October 14, 1914, with an attendance of about fifty.

In recent numbers of *Comptes Rendus* of the French Academy of Science appear the following preliminary notes: May 25, Professor T. H. Gronwall on "The series of Laplace"; June 8, Professor T. H. Gronwall on "Some methods of summation and their application to Fourier's series"; June 15, Professor C. N. Moore on "The relation between certain methods for the summation of a divergent series." The last two are independent announcements to the effect that the method of de La Vallée-Poussin is more general than that of Cesàro.

Among recent publications of interest to teachers of mathematics is "The Teaching of Mathematics" by Professor R. E. Manchester, of the State Normal School, Oshkosh, Wisconsin. This text of 75 pages, published by C. W. Bardeen, Syracuse, after treating the practical benefits of the study of mathematics and the leading considerations in the teaching of arithmetic, algebra, and geometry, deals with the place of mathematics in vocational schools and with methods and modes of presenting these subjects.

The Central Association of Science and Mathematics Teachers met in the Hyde Park High School, Chicago, November 27 and 28. According to the program as issued by the secretary, Mr. W. L. Eikenberry, University High School, University of Chicago, the subject of last year " Vocational courses in science and mathematics " was continued as a part of this year's subject for discussion, viz., " Problems arising in adjusting applied science to high school curricula." A full report of report of the meetings of the Mathematics Section will be made in a later issue.

The Macmillan Co. has recently published " The theory of relativity," a book of 295 pages by L. Silberstein, lecturer in natural philosophy at the University of Rome. It is based, in part, upon a course of lectures delivered in University College, London, in 1912–13.

The British Board of Education has issued a " Memorandum on the Teaching of Geometry in Secondary Schools " which covers essentially the same ground as the circular on geometry issued five years ago by the Board and now out of print.

A large volume entitled " List of Prime Numbers from 1 to 10,006,721 " by Professor D. N. Lehmer, University of California, was recently published by the Carnegie Institution of Washington. The last four digits of the primes are given in the body of the table and the remaining digits are at the top and bottom of the columns. Five thousand primes are listed on each page, and the table gives the rank of any prime in the series of primes. The same precautions were taken in printing these tables as were observed in printing the factor tables for the first ten million numbers, which were prepared in 1909 by the same author under the auspices of the Carnegie Institution. By the publication of these tables Professor Lehmer has rendered an important service to mathematics, in view of their great accuracy and convenient arrangement.

The last volume (XXI) of the *Proceedings* of the Society for the Promotion of Engineering Education contains, among numerous articles of a more special nature, three which have a general interest and which perhaps deserve mention here. The titles are

(1) How Can the Colleges and the Industries Cooperate?
By Ivy L. Lee, Executive Assistant, Penn. R. R. Co.

(2) [Same title as (1)].
By Edward D. Sabine, Terminal Engineer, N. Y. C. & H. R. R.

(3) Recommendations Concerning the Units of Force.
By Professor E. V. Huntington, Harvard University.

The first two will be interesting and instructive to upper classmen in schools of engineering. The third is especially recommended to teachers of physics in secondary schools.

The universities of Great Britain, according to announcements, opened as usual this academic year with probably half the regular attendance, one half being in active military service. The German universities also are reported to have opened.

Professor F. R. Moulton published in *Popular Astronomy* for August–September a very lucid and comprehensive analysis of the works of the noted American astronomer, Dr. George W. Hill, who died April 16, 1914.

"Problemi della Scienza," published in 1906 by F. Enriques, professor of projective and descriptive geometry in the University of Bologna, has been translated into English by Katharine Royce; the book is published by the Open Court Publishing Company under the title "Problems of Science."

A press dispatch appeared in August describing the unusual mental calculations carried out before a class in psychology at the summer session of the University of California by Arthur A. Gamble, "a 21-year-old student of the University of Chicago." A letter of inquiry to Professor Warner Brown of the department of psychology in the University of California brought an authoritative account from which the following interesting extracts are taken.

"He is not a college student.[1] . . . He knows no mathematics beyond simple arithmetic—does not even know what logarithms are or how they are used. He does know several short-cuts in multiplication and division (multiplying from the left, etc.). His *forte* is factoring. He can give offhand (5 seconds or less) two numbers which when multiplied together make any 5- or 6-place number which is proposed; he often gives two numbers which nearly make the given number, with a statement of the remainder. As you see, this depends upon the knowledge of a rather extended multiplication table, say up to 500. He is familiar enough with the products of numbers to be able to give squares and cubes quickly. . . .

"Division he accomplishes by tentative multiplication, which is easier for him. Cube roots are not easy for him; he makes a large number of guesses and tries them out by multiplication. . . . He knows the powers of 2 ("doubling a penny") up to the 32d and can reel them off as fast as he can speak.[2]

"He knows the data necessary for calendar computations and gives the day of the week for any date very quickly.

"He *cannot* add very rapidly.[3] His memory for *digits* is not unusual, but he can hold in memory a large collection of *numbers*. . . . He has an unbounded

[1] He visited the University of Chicago and members of the mathematical staff tried him out with the same conclusions as reached by Professor Brown. They advised him to return to his home in Rochester, N. Y., and continue his regular high school course rather than continue to develop his freakish propensities. EDITORS.

[2] Evidently memorized from the famous horse shoe problem. ED.

[3] The newspaper item stated that Dr. Brown handed simultaneously to Mr. Gamble and an expert adding machine operator a list of numbers to be added, all of four or more digits, Mr. Gamble obtaining the correct sum more quickly than the mechanical adder. ED.

interest in numbers as numbers. The result is that he has come to individualize thousands of them to almost the same extent that you or I individualize the numbers below 100; he knows their relationships in the arithmetical family."

L'Enseignement Mathematique is doubtless the leading journal devoted to the pedagogy of mathematics. This appears under the joint editorship of C. A. Laisant of Paris and H. Fehr of Geneva, with the collaboration of A. Buhl of Toulouse; it is published by Gauthier-Villars of Paris and Georg and Cie of Geneva. The publication is now in the sixteenth year; the subscription price is 15 francs per year (Stechert and Co., West 25th St., New York City). Each volume consists of about 500 pages.

Probably the general nature of the journal is best illustrated by a summary of the articles which appear in the current volume (1914). The first number, issued January 15, 1914, of some 80 pages, includes articles on non-Euclidean geometry as applied to the theory of relativity, on multiple equalities, on the integration of the equations for the movements of a planet about the sun, on some points on the theory of sets, on transcendental plane curves given by equations in which the variables are separable, on a double system of lines on a surface, on an application of the rule of false position, and on new formulas for Heronian triangles. These articles by French, German, Italian, and Russian authors are all written in French. In addition to the articles, space is devoted to current events, to notes and documents, to reviews, and bibliography. The second number is of the same general nature as the first, while the third, fourth, and fifth issues are devoted almost entirely to a report of the International Conference on the teaching of mathematics which was held at Paris, April 1–4, 1914.

The titles of the articles, above-mentioned, show clearly that the appeal is made not to the beginner in mathematics but to the real student of the science. Historical articles are of rather infrequent occurrence. The notes and documents in recent issues have been devoted in large measure to the activities of the International Commission, while the treatment of reviews and bibliography of current literature in the field does not vary greatly from the procedure in American journals.

ERRATA.

Page 164, line 4 up. For J. A. Colson, *read* J. W. Clawson. Page 190, under Algebra. For **416** *read* **416A**, and for **417** *read* **417A**. Page 258, line 8 up. For $2^{2\alpha_3}$ *read* $2^{2^{\alpha_3}}$ Page 259, line 2 down. For $\chi^{p(p-1)}$ *read* $x^{p(p-1)}$. Page 290, line 2 up. For dr^{n-1} *read* $d^{n-1}r$.

Also on page 259, the author wishes to substitute for the paragraph (4) the following: "To point out that although the *Conchoid of Nicomedes* is used in the text to trisect an angle, this application of the curve was a discovery claimed by Pappus (about 300 A. D.).[1] Nicomedes (about 180 B. C.) used the curve for the duplication of the cube[2] and we have only the assertion of Proclus that Nicomedes also used it for the trisection of an angle."

Page 299, last line, last term. For $(-1)^k$ *read* $(-1)^{k-1}$.